Optimal switching and dividend payment with transaction costs

Pablo Azcue and Nora Muler

EAJ - Lyon 2016

Problem: Optimal dividend payment and optimal closing time.

The manager of an insurance company pays dividends to the shareholders and has the possibility of closing a branch of the company at any time. There is a positive transaction cost for any dividend payment and cost for closing the branch.

The decision of closing the branch is irreversible. The problem is to find the time and size of the dividend payments and the time of closing of the branch in order to maximize the expected discounted dividend payment until the ruin time.

There are others problems of optimal impulse and optimal irreversible switching times that can be solved with the same techniques.

Model: Let us assume the uncontrolled surplus processes $X_t^1,\ X_t^2$ of the two branches of an insurance company follow the compound Poisson processes

$$S_t^1 = X_t^1 - X_0^1 = p_1 t - \sum_{i=1}^{N_t^{(1)}} U_i^{(1)} \quad S_t^2 = X_t^2 - X_0^2 = p_2 t - \sum_{j=1}^{N_t^{(2)}} U_j^{(2)}$$

where $N_t^{(1)}$ and $N_t^{(2)}$ are independent Poisson processes with intensities β_1 and β_2 respectively and the claims sizes $U_i^{(1)}$ and $U_j^{(2)}$ are independent with claim-size distributions F_1 and F_2 respectively. So, the uncontrolled joint surplus with initial surplus x also follows a compound Poisson process

$$X_t^0 = x + \underbrace{(p_1 + p_2)}_{p_0} t - \underbrace{\left(\sum_{i=1}^{N_t^{(1)}} U_i^{(1)} + \sum_{j=1}^{N_t^{(2)}} U_j^{(2)}\right)}_{\sum_{i=1}^{N_t^{(0)}} U_i^{(0)}},$$

the intensity of $N_t^{(0)}$ is $\beta_0 = \beta_1 + \beta_2$, the distribution of $U_i^{(0)}$ is $F_0 = \frac{\beta_1 F_1 + \beta_2 F_1}{\beta_1 + \beta_2}$.

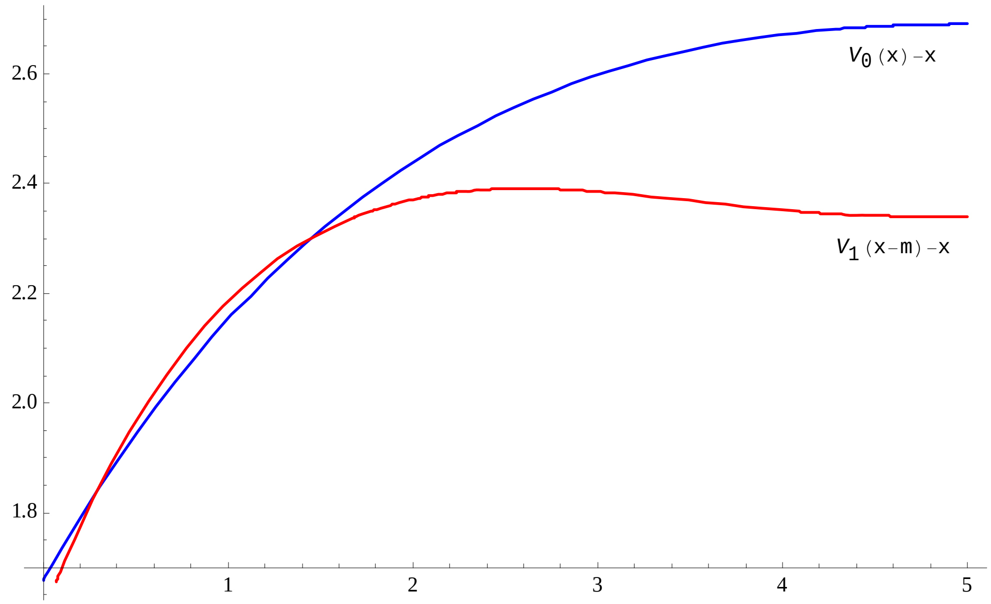
Suppose that the manager of the company has the possibility of closing the second branch at any time $\bar{\tau}$ with a fixed cost m.

Given a initial surplus x, we consider the optimal cumulative discounted dividend payments up to the ruin time in the following cases:

- $V_0(x)$, in the case that we operate with the two branches
- $V_1(x)$, in the case that we operate with only the first branch

If $V_0(x) > V_1(x-m)$ for all $x \geq m$, then it is never optimal to close the second branch; and if $V_0(x) < V_1(x-m)$ for all $x \geq m$, it is optimal to close the second branch immediately.

But in the following situation:



it is not clear whether the optimal time to close the second branch is at any surplus between the two intersection points.

Now, we formulate the problem:

Surplus process

(1) If $t < \bar{\tau}$: $X_t^{\bar{\tau}} = X_t^{(0)} = x + p_0 t - \sum_{i=1}^{N_t^1} U_i^{(0)}$

(2) At the stopping time $\bar{\tau}$ (if $\bar{\tau} < \infty$), the manager close de second branch paying a fixed cost m, so only the first branch survives. For $t \geq \bar{\tau}$:

$$X_t^{\bar{\tau}} = X_{\bar{\tau}}^{(1)} - m + p_1(t - \bar{\tau}) - \sum_{i=N_{\bar{\tau}}^1+1}^{N_t^1} U_i^{(1)}$$

Remark: We allow the closing time $\bar{\tau}$ to be infinity, so it is possible to keep the two branches forever.

Dividend payment:

The manager pays dividends, let L_t be the cumulative dividend payment up to time t : $X_t^{L,\bar{\tau}} = X_t^{\bar{\tau}} - L_t$ and the ruin time is defined as

$$\tau^{L,\bar{\tau}} = \inf\{t \geq 0 : X_t^{L,\bar{\tau}} < 0\}.$$

We assume that, for every dividend payment ξ, a positive transaction cost $K > 0$ has to be paid. This assumption leads to an impulse problem: the amount of dividends L_t paid up to time t should be given by

$$L_t = \sum_{j=1}^{\infty} \xi_j \, I_{\{\tau_j \leq t\}}.$$

Here ξ_j is the dividend payment at time τ_j.

Given a initial surplus x, an admissible dividend payment strategy L_t and a switching time $\overline{\tau}$, the associated expected discounted dividend payment up to the ruin time of this strategy is:

$$V_{L,\overline{\tau}}(x) = E_x\left(\sum_{j=1}^{\infty}\left(\xi_j - K\right)^+ e^{-c\tau_j} I_{\{\tau_j \leq \tau^{L,\overline{\tau}}\}}\right).$$

We define the optimal value function as

$$V(x) = \sup_{(L,\overline{\tau})} V_{L,\overline{\tau}}(x).$$

Also, we ask ourselves if there exists an optimal switching-dividend payment strategy. That is, whether there exists a strategy $\left(L^*, \overline{\tau}^*\right)$ such that

$$V(x) = V_{L^*,\overline{\tau}^*}(x).$$

We first analyze the problem **without closing time**.

Simpler Case: Optimal value function (without closing time).

In the simplest dividends impulse problem (without closing time),

$$V_1(x) = \sup_L V_{L,\infty}(x),$$

we have the following results:

- V_1 is increasing and Lipschitz, that is: there exists a constant $l > 0$ such that $0 \le V_1(b) - V_1(a) \le l(b-a)$ for $b \ge a$
- V_1 is a viscosity solution of the **Hamilton-Jacobi-Bellman** equation $\max\{L_1(V_1)(x), L_2(V_1)(x)\} = 0$ for any $x \ge 0$, where

$$\begin{aligned} L_1(V_1)(x) &= p_1 V_1'(x) - (c+\beta_1)V_1(x) + \beta_1 \int_0^x V_1(x-\alpha)\,dF(\alpha) \\ L_2(V_1)(x) &= \max_{K \le \xi \le x} \left\{ V_1(x-\xi) + (\xi - K) - V_1(x) \right\} \end{aligned}$$

- (Characterization Theorem) V_1 is the **smallest viscosity supersolution** of the HJB equation, that is the smallest solution of $\max\{L_1(V_1)(x), L_2(V_1)(x)\} \leq 0$

  for any $x \geq 0$, where

$$L_1(V_1)(x) = p_1 V_1'(x) - (c + \beta_1)V_1(x) + \beta_1 \int_0^x V_1(x-\alpha)dF(\alpha)$$

$$L_2(V_1)(x) = \max_{K \leq \xi \leq x} \left\{V_1(x - \xi) + (\xi - K) - V_1(x)\right\}$$

- (Verification Theorem) If the value function V^L of a dividend strategy L is a viscosity supersolution of the HJB equation, then $V^L = V_1$.

Optimal strategies in the case without closing time

Given the optimal value function V_1, we consider the operators of the HJB equation

$$\begin{aligned} L_1(V_1)(x) &= p_1 V_1'(x) - (c+\beta_1)V_1(x) + \beta_1 \int_0^x V_1(x-\alpha)dF(\alpha) \\ L_2(V_1)(x) &= \max_{K \le \xi \le x} \left\{ V_1(x-\xi) + (\xi - K) - V_1(x) \right\}, \end{aligned}$$

and the sets

- $A = \{x \geq 0 \text{ s.t. } L_2(V_1)(x) = 0 \text{ and } L_1(V_1)(x) = 0\}$ (set of triggers)
- $B = \{x \geq 0 \text{ s.t. } L_2(V_1)(x) = 0 \text{ and } L_1(V_1)(x) < 0\}$ (set of payments)
- $C = \{x \geq 0 \ s.t. L_2(V_1)(x) < 0\}$ (non-action set)

We have the following properties:

1. A is closed, C is open and B is left-open.

2. The lower boundary of any connected component of B is in A.

3. The upper boundary of any connected component of C is in A.

4. There exists $x^* \in A$ such that $(x^*, \infty) \in B$.

The optimal strategy is:

1. If $x \in C$ pay no dividends up to the exit time of C.
2. If $x \in A$ pay

$$\xi^*(x) = \arg\max_{K \le \xi \le x} \left\{ V_1(x - \xi) + (\xi - K) - V_1(x) \right\}$$

   as dividends.

3. If $x \in B$, let $\widetilde{x} \in A$ be the lower boundary of the connected component of B containing x. Pay $\xi^*(x) = \xi^*(\widetilde{x}) + x - \widetilde{x}$ as dividends. This implies that the surplus $\xi^*(x) - x = \xi^*(\widetilde{x}) - \widetilde{x}$ after a dividend payment is constant in any connected component of B.

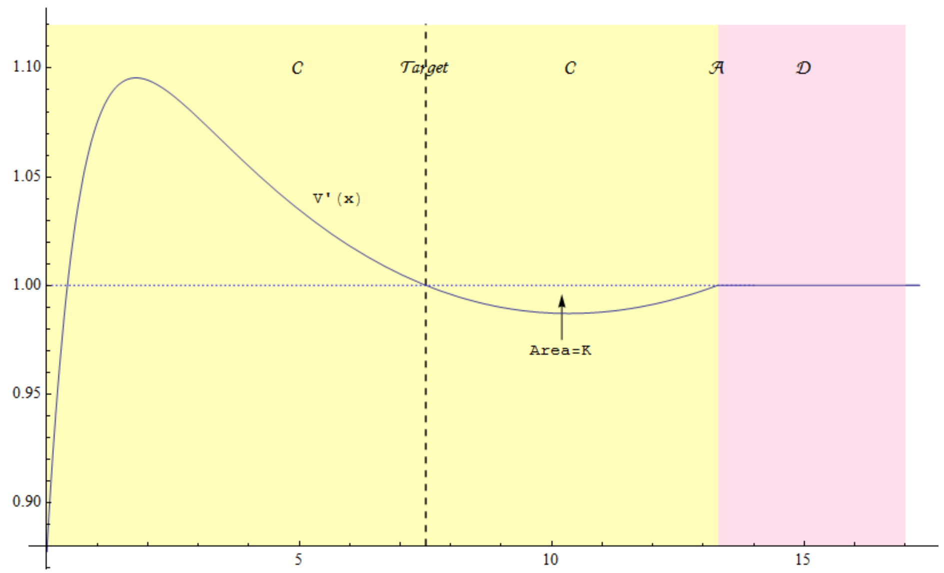
If $x \in A$ we say that x is a **trigger** with **target** $x - \xi^*(x)$. It can be seen that

$$\int_a^b (1 - V_1{}'(x))dx \le K.$$

for any interval $[a, b]$, the pairs trigger/target can be characterized as the smallest intervals where $\int_a^b (1 - V_1{}'(x))dx = K$.

Some examples

In the simplest case the derivative V' (which exists almost everywhere) has this form

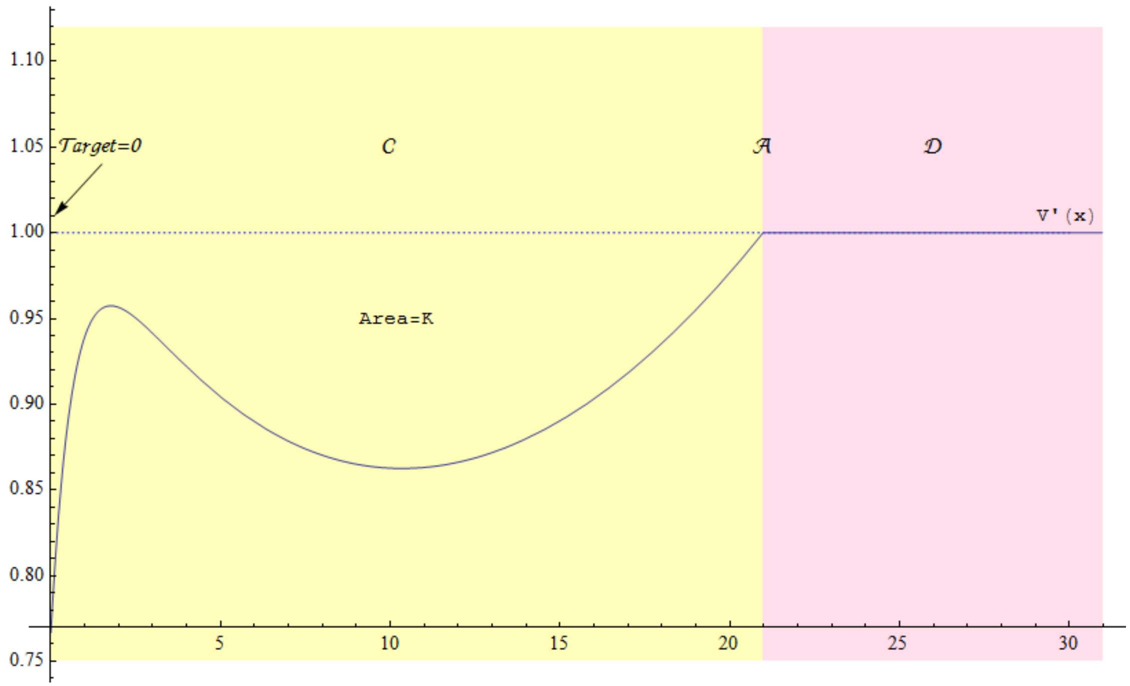


Triggers: b_1, pay $b_1 - a_1$ as dividends (a_1 is the target)

Non-action zone: if $x < b_1$ do not pay any dividend.

Payments zone: (b_1, ∞).

Another example of the simplest case with derivative V'

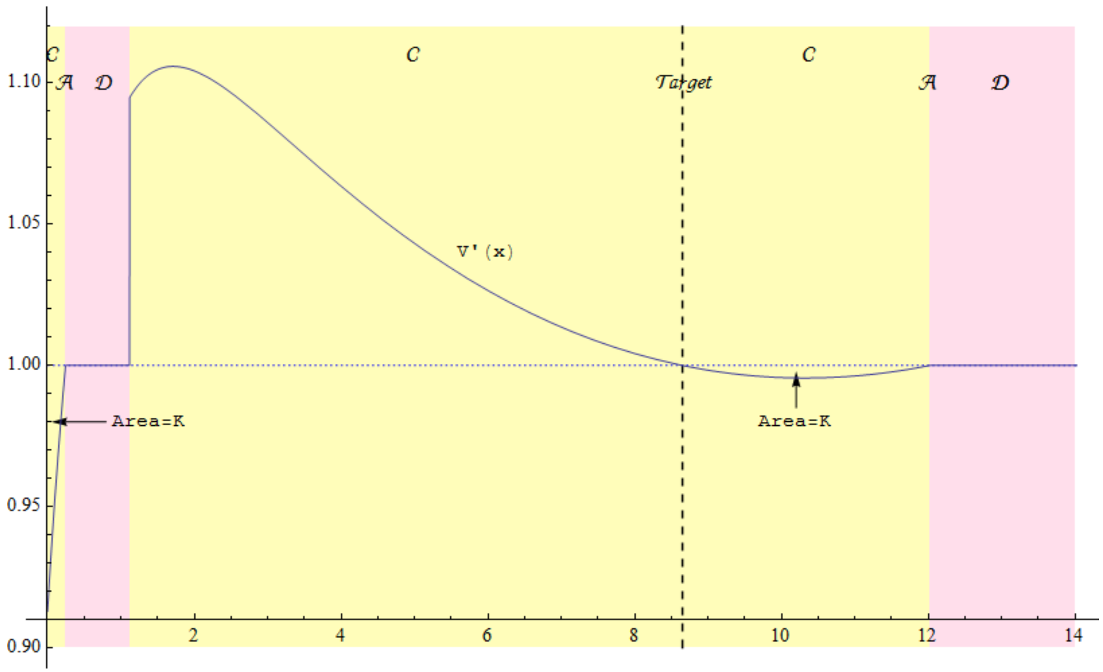


Triggers: b_1, pay $b_1 - a_1$ as dividends (here $a_1 = 0$ is the target)

Non-action zone: if $x < b_1$ do not pay any dividend.

Payments zone: (b_1, ∞). (here $a_1 = 0$ is the target).

Another example of a more complex case with derivative V'



Triggers: b_1 with target $a_1 = 0$ is one target and b_2 with target a_2.

Non-action zone: $(0, b_1) \cup (c_1, b_2)$

Payments zone: (b_1, ∞). (here $a_1 = 0$ is the target)..

Optimal value function in the problem with closing time

V as a solution of an obstacle problem:

Consider $V_1(x)$ the optimal value function corresponding to the optimal expected discounted dividend payment with initial reserve $x \geq 0$ if the uncontrolled reserve of the **first branch alone**:

$$V_1(x) = \sup_L V_L(x) = E_x\left(\int_{0^-}^{\tau^L} e^{-cs} dL_s\right).$$

where τ^L is the ruin time of the process $x + S_t^{(1)} - L_t$. Define $f(x) = V_1(x-m)$.

Proposition:

For any $x \geq 0$,

$$V(x) = \sup_{(L,\bar{\tau})} E_x\left(\int_{0^-}^{\bar{\tau}\wedge\tau^L} e^{-cs} dL_s + e^{-c(\bar{\tau}\wedge\tau^L)} f(X^L_{\bar{\tau}\wedge\tau^L})\right).$$

This formula means that for $t \geq \bar{\tau}$ (after the switching time), it is enough to consider the best dividend payment strategy for the first branch alone and "initial surplus" $X^L_{\bar{\tau}} - m$ (which exists by the results above).

We do not assume that $\bar{\tau}$ is finite, so it is not compulsory to close the second branch.

Theorem: V is the smallest viscosity super-solution of the equation

$$\max\{L_1(V)(x), L_2(V)(x)\} = 0$$

above f.

Equivalently: V is both a solution and the smallest viscosity super-solution of the HJB equation with obstacle

$$\max\{L_1(V)(x), L_2(V)(x), f(x) - V(x)\} = 0.$$

Corollary (verification result): If the value function of a dividend and switching strategy is a viscosity supersolution of the HJB equation with obstacle, then it is the optimal value function.

Optimal strategies in the case without closing time

Given the optimal value function V, we consider the operators

$$
\begin{aligned}
L_1(V)(x) &= pV'(x) - (c+\beta_1)V(x) + \beta_1 \int_0^x V(x-\alpha)\,dF(\alpha) \\
L_2(V)(x) &= \max_{K \le \xi \le x} \left\{ V(x-\xi) + (\xi - K) - V(x) \right\}, \\
L_3(V)(x) &= f(x) - V(x)
\end{aligned}
$$

and the sets

- $D = \{x \geq 0 \text{ s.t.} V(x) = f(x)\}$ (closing zone)
- $A = \{x \geq 0 \text{ s.t. } L_2(V)(x) = L_1(V)(x) = 0 \text{ and } V_1(x) > f(x)\}$ (set of triggers)
- $B = \{x \geq 0 \text{ s.t. } L_2(V)(x) = 0, L_1(V)(x) < 0 \text{ and } V_1(x) > f(x)\}$ (set of dividend payments)
- $C = \{x \geq 0 \text{ s.t. } L_2(V)(x) < 0, L_1(V)(x) = 0 \text{ and } V_1(x) > f(x)\}$ (non-action zone)

We have the following properties:

1. A and D are closed, C is right-open and B is left-open.
2. The lower boundary of any connected component of B is in A.
3. The upper boundary of any connected component of C is in A.
4. There exists $x^* \in A$ such that either $(x^*, \infty) \in B$ or $(x^*, \infty) \in D$

And we define the optimal strategy as:

1. If $x \in D$ close immediately the second branch.
2. If $x \in C$ pay no dividends up to the exit time of C.
3. If $x \in A$ pay

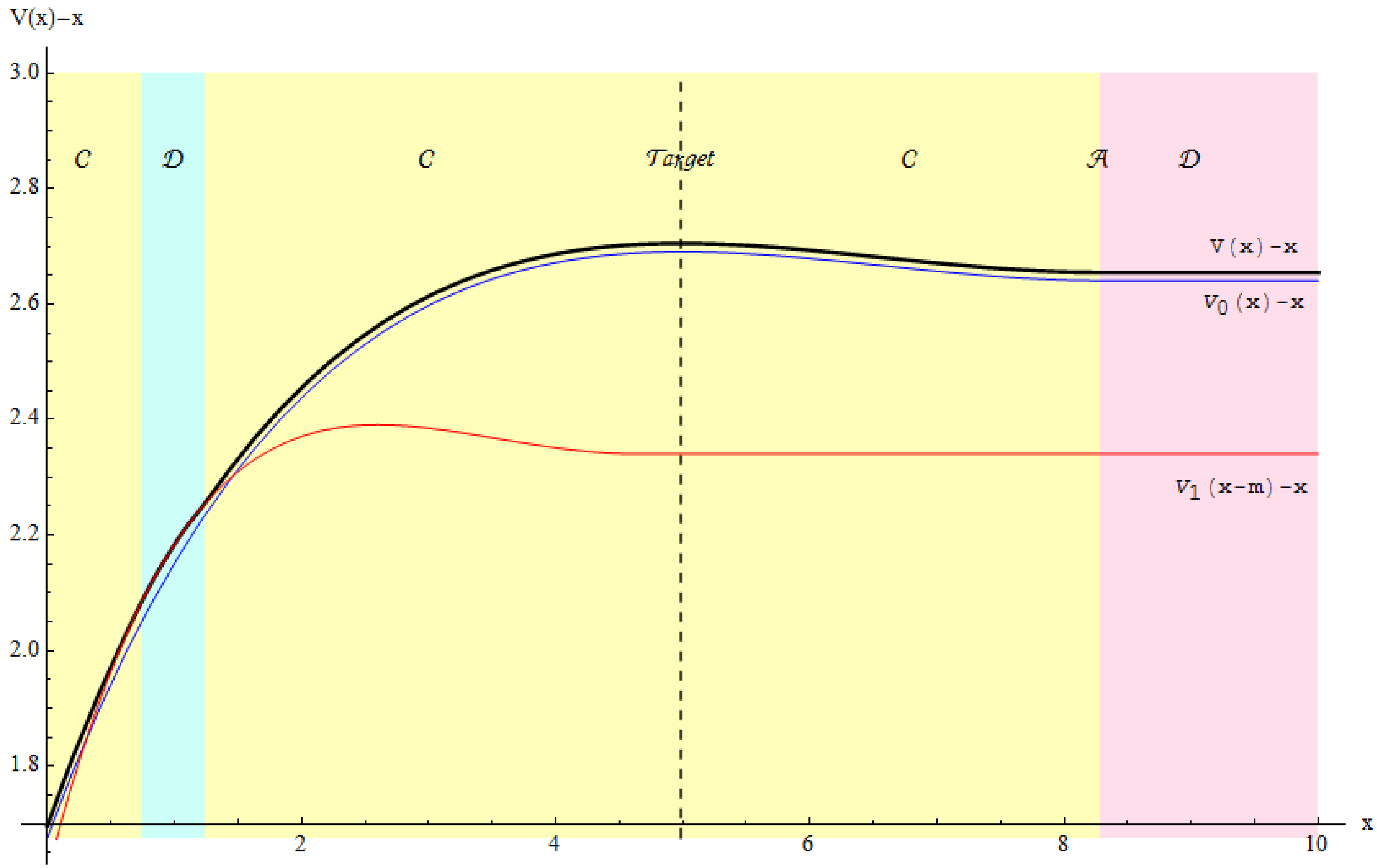
$$\xi^*(x) = \arg \max_{K \leq \xi \leq x} \left\{ V(x - \xi) + (\xi - K) - V(x) \right\}$$

as dividends.

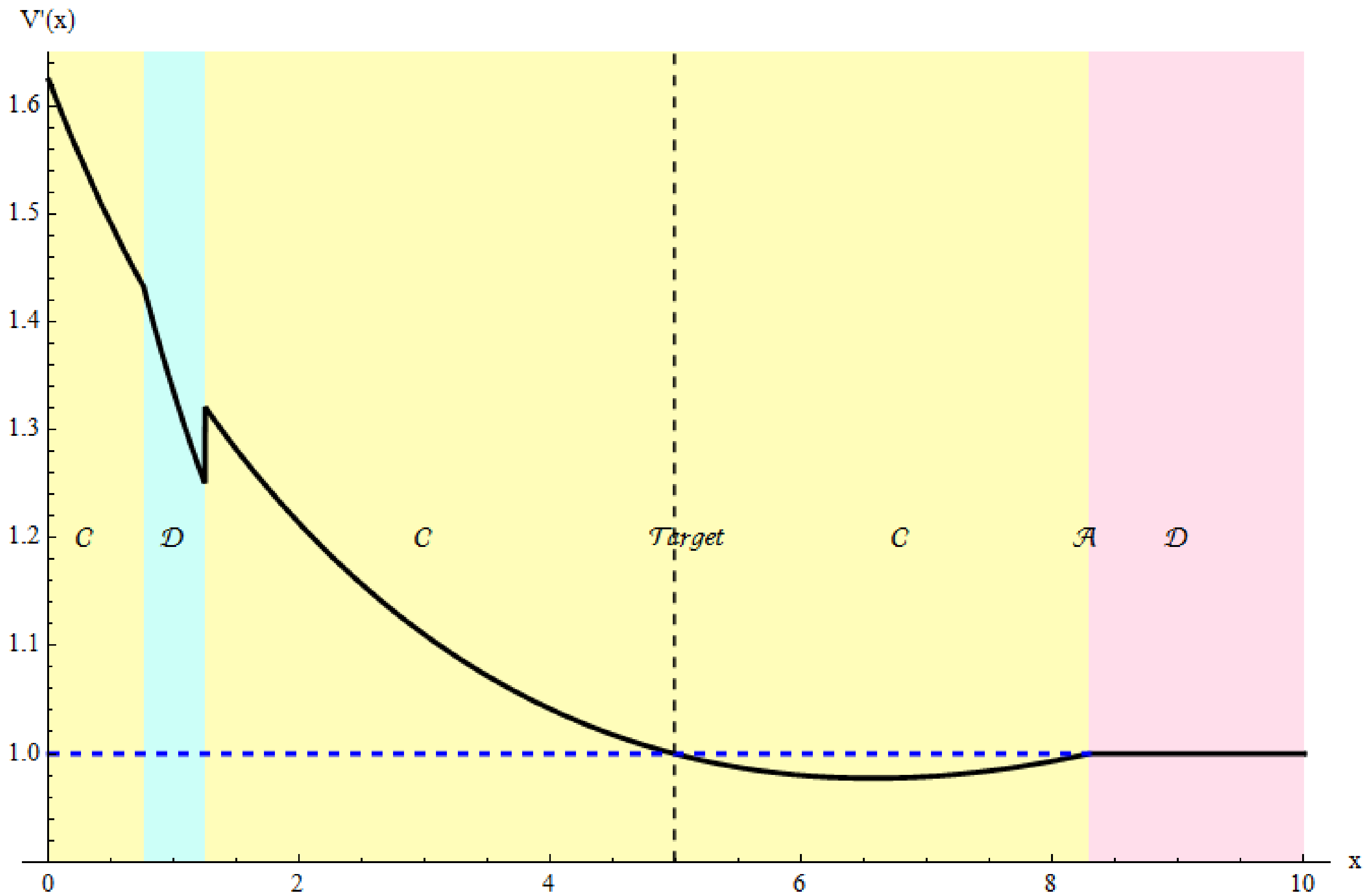
4. If $x \in B$, let $\widetilde{x} \in A$ be the lower boundary of the connected component of B containing x. Pay $\xi^*(x) = \xi^*(\widetilde{x}) + x - \widetilde{x}$ as dividends.

Example of optimal closing time:

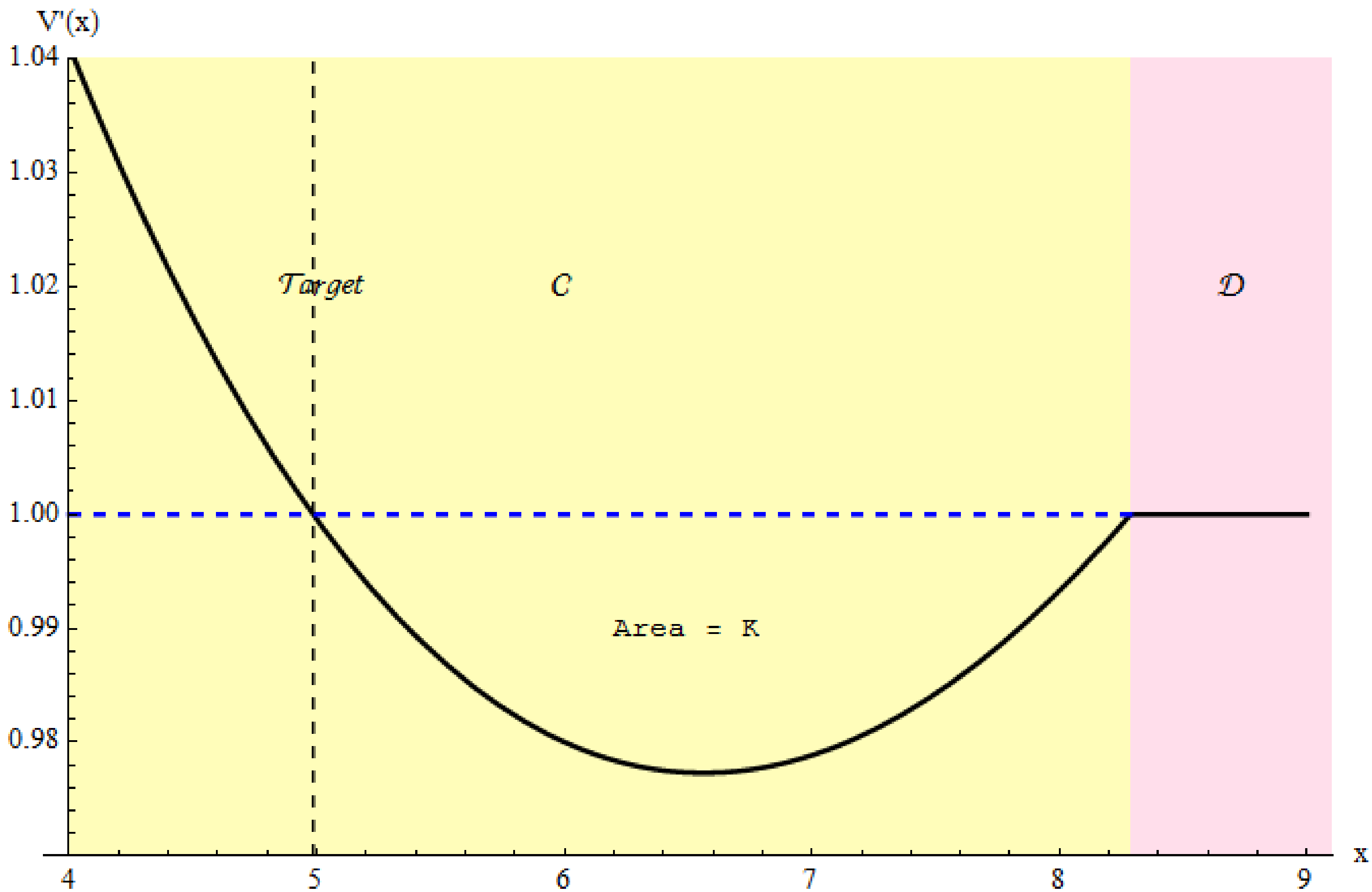
Optimal value function and optimal strategy



Derivative of the optimal value function (1)



Derivative of the optimal value function (2)



Some references.

- For the optimal dividends and switching times problem (no impulse case)

  (a) J.P. Décamps & S. Villeneuve: *Optimal dividend policy and growth option*, Finance Stoch., 11:3–27, 2007.

  (b) Azcue & Muler: *Optimal dividend payment and regime switching in a compound Poisson risk model*. Siam J. Control Optim., 53(5): 3270-3298, 2015.

- For the impulse problem:

  (a) R. Loeffen: *An optimal dividends problem with transaction costs for spectrally negative Lévy processes*. Insurance Math. Econom., 45(1):41–48, 2009

  (b) Thonhauser & Albrecher: *Optimal dividend strategies for a compound Poisson process under transaction cost and power utility*. Stochastic Models, Volume 27, Issue 1, 2011.

Thank you.

Parameters of the problem.

c (impatience rate) $= 0.05$

m (closing cost) $= 0.07$

K (dividends payment cost) $= 0.05$

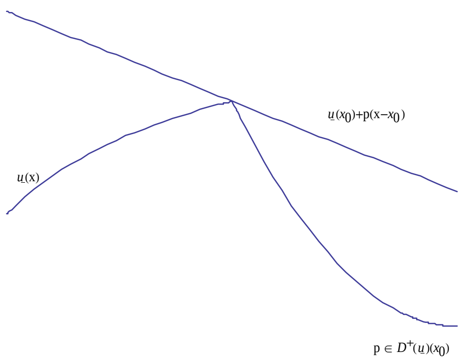
	Branch 1	Branch 2	Branch 1+2
β (claim arrival intensity)	1	2	3
F (claim-size distribution)	$1 - e^{-x}$	$1 - e^{-\frac{3x}{2}}$	$1 - \frac{e^{-x} + e^{-\frac{3x}{2}}}{3}$
η (safety loading)	0.51	0.09	0.195
$p = (1+\eta)\beta\, E(F)$ (premium)	1.007	2.18	3.19

Optimal strategy

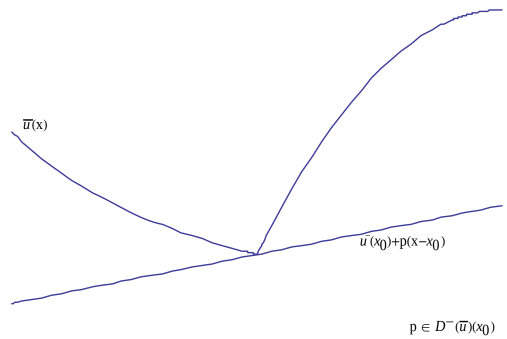
	Branch 1	Branch 2	Branch 1+2 (no closing)	Branch 1+2 (optimal closing)
D (closing zone)	–	–	–	$(0.75, 1.24)$
A (first triggers)	$\{4.54\}$	$\{3.01\}$	$\{8.30\}$	$\{8.28\}$
B (other triggers)	$(4.54, \infty)$	$(3.01, \infty)$	$(8.30, \infty)$	$(8.28, \infty)$
Targets	2.53	0.02	4.99	4.98
C (non-action)	$[0, 4.54)$	$[0, 3.01)$	$[0, 8.30)$	$[0, 0.75) \cup (1.24, 8.28)$

Viscosity Solutions (1).

Given a continuous function u, we say that p is a **super-differential** ($p \in D^{+}(u)(x_0)$) or a **sub-differential** at x_0 ($p \in D^{-}(u)(x_0)$) in the following situations:



super-diff: $p \in D^{+}(u)(x_0)$



sub-diff: $p \in D^{-}(u)(x_0)$

Note that u has both super and sub-differentials at a point x_0 if, and only if, u is differentiable at the point x_0.

Viscosity Solutions (2).

Consider the operator $L(u)(x) = f(x, u(x), u'(x), u(\bullet))$.

The function u is a **viscosity solution** of $L(u(x), u'(x), u(.)) = 0$ at x_0, if for any super-differential $p \in D^+(u)(x_0)$, we have

$$f(x_0, u(x_0), p, u(\bullet)) \geq 0$$

and for any sub-differential $p \in D^-(u)(x_0)$, we have

$$f(x_0, u(x_0), p, u(\bullet)) \leq 0.$$

Note that:

- A classical solution is also a viscosity solution.
- The viscosity solutions of $L(u) = 0$ and $-L(u) = 0$ could be different.