Optimal Stop-loss Reinsurance Strategy under Distortion Risk Measures

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Overview

Preliminaries and notations

Definition (Distortion Risk Measure)

$$\rho^{\Pi}(X) = \int_0^1 \mathrm{VaR}_s(X) d\Pi(s), \tag{1}$$

where Value at Risk is defined as

$$\mathrm{VaR}_{\alpha}(X) = \inf\{x \in \mathbb{R} | \mathbb{P}(X \leq x) \geq \alpha\} \tag{2}$$

and Π:$[0, 1] \to [0, 1]$ *is a distortion function which is a non-decreasing and cádlág function.*

Examples of Distortion Risk Measures

- Value at Risk (VaR) at the confidence level α with distortion function $\Pi(u) = \mathbf{1}_{[\alpha,1]}(u)$.
- Conditional Value at Risk (CVaR) with distortion function $\Pi(u) = \frac{u-\alpha}{1-\alpha}\mathbf{1}_{[\alpha,1]}(u)$ and distortion form as

$$\mathrm{CVaR}_\alpha(X) = \frac{1}{1-\alpha}\int_\alpha^1 \mathrm{VaR}_s(X)ds \tag{3}$$

- Wang's Premium with Wang's transformation $g_\gamma(u) = \Phi(\Phi^{-1}(u) + \gamma)$ as the distortion function, where $\gamma \in \mathbb{R}$ is a real parameter and Φ is the cumulative distribution function of standard Normal distribution. Use the properties of Φ, the dual distortion funcion can be written as $\Pi_\gamma(u) = 1 - g_\gamma(1-u) = \Phi(\Phi^{-1}(u) - \gamma)$

Preliminaries and notations

Assumptions

1. $F_{X_{t+1}}(.)$ is the strictly increasing cdf of total claims X_{t+1} from $(t, t+1]$ and X_i are i.i.d.
2. The shareholds' benifits is the first priority then consider on both cedent's point of view and the pressure from the reinsurance company.

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Capped Stop-loss Reinsurance

The claims covered by the capped stop-loss reinsurance contract are

$$R_{a_t}(X_{t+1}) = \begin{cases} X_{t+1}, & X_{t+1} < a_t \\ a_t, & X_{t+1} \geq a_t \end{cases}$$

where a_t is the retention level decided at time t covering the total claims occuring in $(t, t+1]$.

Preliminaries and notations

Discrete Time 'Surplus' Model

U_{t+1} is the balance of the insurance company before deciding the reinsurance retention level which covers the claims of next period and paying out the dividends,i.e.,

$$U_{t+1} = \frac{U_t - D_t}{\beta} + c - (X_{t+1} - R_{a_t}(X_{t+1})) - \rho^{\Pi}[R_{a_t}(X_{t+1})] \quad (4)$$

for $t = 0, 1, 2, \cdots$, where D_t is the dividends paid at time t, $\beta = \frac{1}{1+r}$ is a dicounted factor and r is the interest rate per unit time.

Preliminaries and notations

Barrier Dividend Policy

The barrier dividend policy at time $t+1$ is set by

$$D_{t+1} = (U_{t+1} - b)_+ =: H[U_{t+1}] \tag{5}$$

where $(U_{t+1} - b)_+ = \max\{U_{t+1} - b, 0\}$, $b \in \mathbb{R}$ is the known constant dividend barrier.

Notice:

Equation(4) $\Leftrightarrow$

$$U_{t+1} = \frac{U_t - H[U_t]}{\beta} + c - (X_{t+1} - R_{a_t}(X_{t+1})) - \rho^{\Pi}[R_{a_t}(X_{t+1})] \tag{6}$$

Optimization Without Solvency Condition

Static Case

The objective function is

$$\max_a \mathbb{E}\left[H(U_1)\right] = \max_a \mathbb{E}\left[(U_1 - b)_+\right] \tag{7}$$

Denote $K := \frac{u_0 - H(u_0)}{\beta} + c - b \in \mathbb{R}$ so the discrete time surplus can be rewritten as

$$U_1 = K + b - (X - a)_+ - \rho^{\Pi}[R_a(X)] \tag{8}$$

And the objective function can be simplified in the static model as

$$\max_a \mathbb{E}\left[\left(K - (X - a)_+ - \rho^{\Pi}[R_a(X)]\right)_+\right] \tag{9}$$

The expectation now can be rewritten in the form of integrals as

$$\mathbb{E}\left[(K-(X-a)_+-\rho^{\Pi}[R_a(X)])_+\right]$$
$$=(\mathbf{1}_{\{K>\rho^{\Pi}[R_a(X)]\}})[\int_0^a (K-\rho^{\Pi}[R_a(X)])dF_X(x)$$
$$+\int_a^{K+a-\rho^{\Pi}[R_a(X)]}(K-(X-a)-\rho^{\Pi}[R_a(X)])dF_X(x)]$$

1. When $K > \rho^{\Pi}[R_a(X)]$ and

$$\frac{\partial}{\partial a}\mathbb{E}\left[(K-(X-a)_+-\rho^{\Pi}[R_a(X)])_+\right]=0 \qquad (10)$$

   We have
   $F_X(a)=\Pi(F_X(a))F_X(a+\frac{u_0-H(u_0)}{\beta}+c-b-\rho^{\Pi}[R_a(X)])$

2. When $K \leq \rho^{\Pi}[R_a(X)]$, $\mathbb{E}[H(U_1)]=0$. The optimal stop-loss retention level, a, should satisfy

$$\rho^{\Pi}[R_a(X)]=\frac{u_0-H(u_0)}{\beta}+c-b \qquad (11)$$

Optimization Without Solvency Condition

Dynamic Case

The objective function is

$$\max_{\{a_s\}_{s=t}^{\infty}} \mathbb{E}_t \left[\sum_{s=t}^{\infty} \beta^{s-t} H(U_s) \right] \tag{12}$$

Subject to the budget constraint:

$$U_{t+1} = \frac{U_t - H(U_t)}{\beta} + c - (X_{t+1} - R_{a_t}(X_{t+1})) - \rho^{\Pi}[R_{a_t}(X_{t+1})] \tag{13}$$

Since we have the simiar framwork as in Section 3.2 of the book by Ljungqvist and Sargent (2004), let the value function be

$$
\begin{aligned}
V(U_t) &= \max_{\{a_s\}_{s=t}^{\infty}} \mathbb{E}_t \left[\sum_{s=t}^{\infty} \beta^{s-t} H(U_{s+1}) \right] \\
&= \max_{a_t} \mathbb{E}_t \left[H(U_t) + \max_{\{a_s\}_{s=t+1}^{\infty}} \mathbb{E}_{t+1} \left[\sum_{s=t+1}^{\infty} \beta^{s-t} H(U_{s+1}) \right] \right] \\
&= \max_{a_t} \{ H(U_t) + \beta \mathbb{E}_t \left[V(g[U_t, a_t; X_{t+1}]) \right] \}
\end{aligned}
$$

i.e.

$$
V(u) = \max_{a} \{ H(u) + \beta \mathbb{E}_t \left[V(g[u, a; X]) \right] \} \tag{14}
$$

Take the partial differential with respect to the control variable a

$$0 = \beta \mathbb{E}_t \left[V'(g[u,a;X]) \frac{\partial g[u,a;X]}{\partial a} \right] \tag{15}$$

$$\Leftrightarrow 0 = \mathbb{E}_t \left[V'(g[u,a;X])(\mathbf{1}_{\{X \geq a\}} - [1 - \Pi(F_X(a))]) \right] \tag{16}$$

Assume that $a_t = h(U_t)$ which means that the objective policy is a function of the corresponding surplus. Note $h(.)$ is a deterministic function. Thus,

$$V(u) = \max_a \{ H(u) + \beta \mathbb{E}_t \left[V(g[u,h(u);X]) \right] \}$$

Apply the Envelope Theorem and differentiate it with respect to u to obtain

$$V'(u) = H'(u)$$
$$+ \beta \mathbb{E}_t \left[V'(g[u,h(u);X]) \left(\frac{\partial g[u,h(u);X]}{\partial u} + \frac{\partial g[u,h(u);X]}{\partial h} \frac{\partial h(u)}{\partial u} \right) \right]$$

Follow the work from Benveniste and Scheinkman (1979), we assume that next period surplus can be rewritten as $U_{t+1} = g[h(u); X]$. Therefore,

$$V'(u) = H'(u) + \beta \mathbb{E}_t \left[V'(g) \frac{\partial g[h(u); X]}{\partial h} \frac{\partial h(u)}{\partial u} \right] \tag{17}$$

Use the first order condition in equation (15) and apply it to latter period surplus, we have

$$V'(u) = H'(u) \Leftrightarrow V'(g) = H'(g) \tag{18}$$

Take it back to the equation(16), we have

$$\begin{aligned} 0 &= \mathbb{E}_t \left[H'(g[u, a; X])(\mathbf{1}_{\{X \geq a\}} - [1 - \Pi(F_X(a))]) \right] \\ &= \mathbb{E}_t \left[\mathbf{1}_{\{g \geq b\}}(\mathbf{1}_{\{X \geq a\}} - [1 - \Pi(F_X(a))]) \right] \end{aligned}$$

Equivalently,

$$\mathbb{P}[g \geq b, X \geq a | u][\Pi(F_X(a))] = \mathbb{P}[g \geq b, X < a | u][1 - \Pi(F_X(a))] \tag{19}$$

For the conditional probability on the **left hand side**,

$$\begin{aligned}&\mathbb{P}[g \geq b, X \geq a|u]\\=&\mathbb{P}[a \leq X \leq a + \frac{u - H(u)}{\beta} + c - \rho^{\Pi}[R_a(X)] - b]\\=&F_X(\frac{u - H(u)}{\beta} + c + a - \rho^{\Pi}[R_a(X)] - b) - F_X(a)\end{aligned}$$

where $\frac{u-H(u)}{\beta} + c - \rho^{\Pi}[R_a(X)] - b \geq 0$, otherwise, $\mathbb{P}[g \geq b, X \geq a|u] = 0$.

For the conditional probability on the **right hand side**,

$$\begin{aligned}&\mathbb{P}[g \geq b, X < a|u]\\=&\mathbb{P}[\frac{u - H(u)}{\beta} + c - \rho^{\Pi}[R_a(X)] - b \geq 0, X < a]\end{aligned}$$

Cosider two cases here.

1. If $\frac{u-H(u)}{\beta} + c - \rho^{\Pi}[R_a(X)] - b > 0$, $\mathbb{P}[g \geq b, X < a|u] = F_X(a)$ and

$$F_X(a) = F_X(a + \frac{u - H(u)}{\beta} + c - \rho^{\Pi}[R_a(X)] - b)[\Pi(F_X(a))] \tag{20}$$

2. If $\frac{u-H(u)}{\beta} + c - \rho^{\Pi}[R_a(X)] - b \leq 0$, $\mathbb{P}[g \geq b, X \geq a|u] = \mathbb{P}[g \geq b, X < a|u] = 0$. Therefore, $\mathbb{P}[g \geq b|u] = \mathbb{P}[g \geq b, X \geq a|u] + \mathbb{P}[g \geq b, X < a|u] = 0$ which means that knowing the information at time t, there is no possibility of paying dividends in the next period. The optimal retention level needs to satisfy

$$\rho^{\Pi}[R_a(X)] = \frac{u - H(u)}{\beta} + c - b \tag{21}$$

Myopic Policy

Optimization With Solvency Condition

Solvency Condition

$$0 \leq \rho^{\Gamma}(-U_{t+1})$$
$$= -(\frac{U_t - H(U_t)}{\beta} - \rho^{\Pi}_{a_t}(X_{t+1}) - \rho^{\Gamma}(X_{t+1} - R_{a_t}(X_{t+1})) + c)$$

Brief Steps to the Results

Using Lagrangian multiplier λ, Subject to

$$\lambda\left[\rho^{\Pi}[R_a(X)] + \rho^{\Gamma}(X) - \rho^{\Gamma}[R_a(X)] - K - b\right] = 0$$

$$\lambda \geq 0$$

Brief Steps to the Results

Using Lagrangian multiplier λ, Subject to

$$\lambda\left[\rho^{\Pi}[R_a(X)] + \rho^{\Gamma}(X) - \rho^{\Gamma}[R_a(X)] - K - b\right] = 0$$
$$\lambda \geq 0$$

1. $\lambda = 0$, the same results in Theorem 8.
2. $\lambda > 0$ and $K \leq \rho^{\Pi}[R_a(X)]$, $\Gamma(F_X(a)) = \Pi(F_X(a))$ need to be held.
3. $\lambda > 0$ and $K > \rho^{\Pi}[R_a(X)]$, $\lambda = \dfrac{\Pi(F_X(a))F_X(K + a - \rho^{\Pi}[R_a(X)]) - F_X(a)}{\Gamma(F_X(a)) - \Pi(F_X(a))}$ when $\Gamma(F_X(a)) \neq \Pi(F_X(a))$; or both $\Pi(F_X(a))F_X(K + a - \rho^{\Pi}[R_a(X)]) = F_X(a)$ and $\Gamma(F_X(a)) = \Pi(F_X(a))$ holds.

Example under Value at Risk

Consider both distortion risk measures for solvency condition and reinsurance premium as Value at Risk with different confidence level γ and α respectively, i.e., $\rho^{\Gamma}(.) = \text{VaR}_{\gamma}(.)$ and $\rho^{\Pi}(.) = \text{VaR}_{\alpha}(.)$, the corresponding distortion functions are $\Gamma(u) = \mathbf{1}_{[\gamma,1]}(u)$ and $\Pi(u) = \mathbf{1}_{[\alpha,1]}(u)$.

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1. Check if $K(u, c, b) > 0$, then $a^* = 0$ while $\max \mathbb{E}[H(U_1)] = \int_0^K F_X(x)dx$; Otherwise, go to step 2.
2. Check if $F_X^{-1}(\alpha) = K$ or $F_X^{-1}(\alpha) = K + b$, then $a^* \in [\text{VaR}_{\alpha\vee\gamma}(X), \infty)$ while $\max \mathbb{E}[H(U_1)] = 0$; Otherwise, go to step 3.
3. Check if $F_X^{-1}(\gamma) = K + b$, then $a^* \in [K, \text{VaR}_{\alpha\wedge\gamma}(X)]$ while $\max \mathbb{E}[H(U_1)] = 0$; Otherwise, $a^* \in [K, \infty)$ while $\max \mathbb{E}[H(U_1)] = 0$.

Bibliography

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Thank you for Your Attention.