Optimal Stop-loss Reinsurance Strategy under Distortion Risk Measures

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Preliminaries and notations

Definition (Distortion Risk Measure)

\[ \rho^\Pi(X) = \int_0^1 \text{VaR}_s(X) d\Pi(s), \]  

where Value at Risk is defined as

\[ \text{VaR}_\alpha(X) = \inf\{x \in \mathbb{R} | \mathbb{P}(X \leq x) \geq \alpha \} \]  

and \( \Pi:[0,1] \rightarrow [0,1] \) is a distortion function which is a non-decreasing and cádlág function.
Examples of Distortion Risk Measures

- Value at Risk (VaR) at the confidence level $\alpha$ with distortion function $\Pi(u) = 1_{[\alpha,1]}(u)$.
- Conditional Value at Risk (CVaR) with distortion function $\Pi(u) = \frac{u - \alpha}{1 - \alpha} 1_{[\alpha,1]}(u)$ and distortion form as
  \[
  \text{CVaR}_\alpha(X) = \frac{1}{1 - \alpha} \int_{\alpha}^{1} \text{VaR}_s(X) \, ds
  \]  

- Wang’s Premium with Wang’s transformation $g_\gamma(u) = \Phi(\Phi^{-1}(u) + \gamma)$ as the distortion function, where $\gamma \in \mathbb{R}$ is a real parameter and $\Phi$ is the cumulative distribution function of standard Normal distribution. Use the properties of $\Phi$, the dual distortion function can be written as $\Pi_\gamma(u) = 1 - g_\gamma(1 - u) = \Phi(\Phi^{-1}(u) - \gamma)$
Preliminaries and notations

Assumptions

1. $F_{X_{t+1}}(.)$ is the strictly increasing cdf of total claims $X_{t+1}$ from $(t, t + 1]$ and $X_i$ are i.i.d.

2. The shareholders’ benefits is the first priority then consider on both cedent’s point of view and the pressure from the reinsurance company.
Preliminaries and notations

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Capped Stop-loss Reinsurance

The claims covered by the capped stop-loss reinsurance contract are

$$R_{a_t}(X_{t+1}) = \begin{cases} X_{t+1}, & X_{t+1} < a_t \\ a_t, & X_{t+1} \geq a_t \end{cases}$$

where $a_t$ is the retention level decided at time $t$ covering the total claims occurring in $(t, t + 1]$. 

Preliminaries and notations

Discrete Time 'Surplus' Model

$U_{t+1}$ is the balance of the insurance company before deciding the reinsurance retention level which covers the claims of next period and paying out the dividends, i.e.,

$$U_{t+1} = \frac{U_t - D_t}{\beta} + c - (X_{t+1} - R_a(X_{t+1})) - \rho \Pi [R_a(X_{t+1})] \quad (4)$$

for $t = 0, 1, 2, \cdots$, where $D_t$ is the dividends paid at time $t$, $\beta = \frac{1}{1+r}$ is a discounted factor and $r$ is the interest rate per unit time.
**Preliminaries and notations**

**Barrier Dividend Policy**

The barrier dividend policy at time \( t + 1 \) is set by

\[
D_{t+1} = (U_{t+1} - b)_+ =: H[U_{t+1}]
\]

where \((U_{t+1} - b)_+ = \max\{U_{t+1} - b, 0\}, \ b \in \mathbb{R}\) is the known constant dividend barrier.

**Notice:**

Equation (4) \iff

\[
U_{t+1} = \frac{U_t - H[U_t]}{\beta} + c - (X_{t+1} - R_a(X_{t+1})) - \rho \Pi[R_a(X_{t+1})]
\]

(6)
Optimization Without Solvency Condition

Static Case

The objective function is

$$\max_a \mathbb{E} [H(U_1)] = \max_a \mathbb{E} [(U_1 - b)_+]$$ (7)

Denote $K := \frac{u_0 - H(u_0)}{\beta} + c - b \in \mathbb{R}$ so the discrete time surplus can be rewritten as

$$U_1 = K + b - (X - a)_+ - \rho \Pi [R_a(X)]$$ (8)

And the objective function can be simplified in the static model as

$$\max_a \mathbb{E} \left[ (K - (X - a)_+ - \rho \Pi [R_a(X)])_+ \right]$$ (9)
The expectation now can be rewritten in the form of integrals as

\[
\mathbb{E} \left[ (K - (X - a) \mathbbm{1}\{K > \rho \Pi[R_a(X)]\})^+ \right] \\
= (1\{K > \rho \Pi[R_a(X)]\}) \left[ \int_0^a (K - \rho \Pi[R_a(X)])dF_X(x) \right. \\
+ \int_a^{K+a-\rho \Pi[R_a(X)]} (K - (X - a) - \rho \Pi[R_a(X)])dF_X(x) \\
+ \int_a^{K+a} (K - (X - a) - \rho \Pi[R_a(X)])dF_X(x) \right]
\]
When $K > \rho \Pi[R_a(X)]$ and

$$\frac{\partial}{\partial a} \mathbb{E} \left[ (K - (X - a)_+ - \rho \Pi[R_a(X)])_+ \right] = 0 \quad (10)$$

We have

$$F_X(a) = \Pi(F_X(a))F_X(a + \frac{u_0 - H(u_0)}{\beta} + c - b - \rho \Pi[R_a(X)])$$

When $K \leq \rho \Pi[R_a(X)]$, $\mathbb{E}[H(U_1)] = 0$. The optimal stop-loss retention level, $a$, should satisfy

$$\rho \Pi[R_a(X)] = \frac{u_0 - H(u_0)}{\beta} + c - b \quad (11)$$
Optimization Without Solvency Condition

Dynamic Case

The objective function is

$$\max_{\{a_s\}_{s=t}^{\infty}} \mathbb{E}_t \left[ \sum_{s=t}^{\infty} \beta^{s-t} H(U_s) \right]$$

(12)

Subject to the budget constraint:

$$U_{t+1} = \frac{U_t - H(U_t)}{\beta} + c - (X_{t+1} - R_at(X_{t+1})) - \rho \Pi [R_at(X_{t+1})]$$

(13)
Since we have the similar framework as in Section 3.2 of the book by Ljungqvist and Sargent (2004), let the value function be

\[
V(U_t) = \max_{\{a_s\}_{s=t}^{\infty}} \mathbb{E}_t \left[ \sum_{s=t}^{\infty} \beta^{s-t} H(U_{s+1}) \right]
\]

\[
= \max_{a_t} \mathbb{E}_t \left[ H(U_t) + \max_{\{a_s\}_{s=t+1}^{\infty}} \mathbb{E}_{t+1} \left[ \sum_{s=t+1}^{\infty} \beta^{s-t} H(U_{s+1}) \right] \right]
\]

\[
= \max_{a_t} \left\{ H(U_t) + \beta \mathbb{E}_t \left[ V(g[U_t, a_t; X_{t+1}]) \right] \right\}
\]

i.e.

\[
V(u) = \max_{a} \left\{ H(u) + \beta \mathbb{E}_t \left[ V(g[u, a; X]) \right] \right\}
\]

(14)
Take the partial differential with respect to the control variable $a$

$$0 = \beta \mathbb{E}_t \left[ V'(g[u, a; X]) \frac{\partial g[u, a; X]}{\partial a} \right]$$

$$\Leftrightarrow 0 = \mathbb{E}_t \left[ V'(g[u, a; X]) \left( 1_{\{X \geq a\}} - [1 - \pi(F_X(a))] \right) \right]$$

Assume that $a_t = h(U_t)$ which means that the objective policy is a function of the corresponding surplus. Note $h(.)$ is a deterministic function. Thus,

$$V(u) = \max_a \{ H(u) + \beta \mathbb{E}_t [V(g[u, h(u); X])] \}$$

Apply the Envelope Theorem and differentiate it with respect to $u$ to obtain

$$V'(u) = H'(u) + \beta \mathbb{E}_t \left[ V'(g[u, h(u); X]) \left( \frac{\partial g[u, h(u); X]}{\partial u} + \frac{\partial g[u, h(u); X]}{\partial h} \frac{\partial h(u)}{\partial u} \right) \right]$$
Follow the work from Benveniste and Scheinkman (1979), we assume that next period surplus can be rewritten as $U_{t+1} = g[h(u); X]$. Therefore,

$$V'(u) = H'(u) + \beta \mathbb{E}_t \left[ V'(g) \frac{\partial g[h(u); X]}{\partial h} \frac{\partial h(u)}{\partial u} \right] \quad (17)$$

Use the first order condition in equation (15) and apply it to latter period surplus, we have

$$V'(u) = H'(u) \iff V'(g) = H'(g) \quad (18)$$

Take it back to the equation (16), we have

$$0 = \mathbb{E}_t \left[ H'(g[u, a; X]) \left( \mathbf{1}\{X \geq a\} - [1 - \Pi(F_X(a))] \right) \right]
= \mathbb{E}_t \left[ \mathbf{1}\{g \geq b\} \left( \mathbf{1}\{X \geq a\} - [1 - \Pi(F_X(a))] \right) \right]
$$

Equivalently,

$$\mathbb{P}[g \geq b, X \geq a|u][\Pi(F_X(a))] = \mathbb{P}[g \geq b, X < a|u][1 - \Pi(F_X(a))] \quad (19)$$
For the conditional probability on the left hand side,

\[ P[g \geq b, X \geq a | u] \]

\[ = \mathbb{P}[a \leq X \leq a + \frac{u - H(u)}{\beta} + c - \rho \Pi [R_a(X)] - b] \]

\[ = F_X \left( \frac{u - H(u)}{\beta} + c + a - \rho \Pi [R_a(X)] - b \right) - F_X (a) \]

where \( \frac{u - H(u)}{\beta} + c - \rho \Pi [R_a(X)] - b \geq 0 \), otherwise,

\[ P[g \geq b, X \geq a | u] = 0. \]

For the conditional probability on the right hand side,

\[ P[g \geq b, X < a | u] \]

\[ = P \left[ \frac{u - H(u)}{\beta} + c - \rho \Pi [R_a(X)] - b \geq 0, X < a \right] \]
Consider two cases here.

1. If \( \frac{u - H(u)}{\beta} + c - \rho \Pi[R_a(X)] - b > 0 \),
   \[ P[g \geq b, X < a | u] = F_X(a) \] and
   \[ F_X(a) = F_X(a + \frac{u - H(u)}{\beta} + c - \rho \Pi[R_a(X)] - b) \Pi(F_X(a))] \] \( (20) \)

2. If \( \frac{u - H(u)}{\beta} + c - \rho \Pi[R_a(X)] - b \leq 0 \),
   \[ P[g \geq b, X \geq a | u] = P[g \geq b, X < a | u] = 0. \] Therefore,
   \[ P[g \geq b | u] = P[g \geq b, X \geq a | u] + P[g \geq b, X < a | u] = 0 \]
   which means that knowing the information at time \( t \), there is no possibility of paying dividends in the next period. The optimal retention level needs to satisfy
   \[ \rho \Pi[R_a(X)] = \frac{u - H(u)}{\beta} + c - b \] \( (21) \)
Myopic Policy
Optimization With Solvency Condition

Solvency Condition

\[
0 \leq \rho^\Gamma (\frac{-U_{t+1}}{\beta}) = \frac{U_t - H(U_t)}{\beta} - \rho_{a_t}(X_{t+1}) - \rho^\Gamma(X_{t+1} - R_{a_t}(X_{t+1})) + c
\]
Brief Steps to the Results

Using Lagrangian multiplier $\lambda$, Subject to

$$
\lambda \left[ \rho \Gamma \left[ R_a(X) \right] + \rho \Gamma (X) - \rho \Gamma \left[ R_a(X) \right] - K - b \right] = 0
$$

$$
\lambda \geq 0
$$
Brief Steps to the Results

Using Lagrangian multiplier $\lambda$, Subject to

$$\lambda \left[ \rho \Pi[R_a(X)] + \rho \Gamma(X) - \rho \Gamma[R_a(X)] - K - b \right] = 0$$

$\lambda \geq 0$

1. $\lambda = 0$, the same results in Theorem 8.

2. $\lambda > 0$ and $K \leq \rho \Pi[R_a(X)]$, $\Gamma(F_X(a)) = \Pi(F_X(a))$ need to be held.

3. $\lambda > 0$ and $K > \rho \Pi[R_a(X)]$,

$$\lambda = \frac{\Pi(F_X(a))F_X(K + a - \rho \Pi[R_a(X)]) - F_X(a)}{\Gamma(F_X(a)) - \Pi(F_X(a))} \text{ when }$$

$\Gamma(F_X(a)) \neq \Pi(F_X(a))$; or both

$\Pi(F_X(a))F_X(K + a - \rho \Pi[R_a(X)]) = F_X(a)$ and

$\Gamma(F_X(a)) = \Pi(F_X(a))$ holds.
Example under Value at Risk

Consider both distortion risk measures for solvency condition and reinsurance premium as Value at Risk with different confidence level $\gamma$ and $\alpha$ respectively, i.e., $\rho^\Gamma(.) = \text{VaR}_\gamma(.)$ and $\rho^\Pi(.) = \text{VaR}_\alpha(.)$, the corresponding distortion functions are $\Gamma(u) = 1_{[\gamma,1]}(u)$ and $\Pi(u) = 1_{[\alpha,1]}(u)$. 
Consider both distortion risk measures for solvency condition and reinsurance premium as Value at Risk with different confidence level $\gamma$ and $\alpha$ respectively, i.e., $\rho^\Gamma(.) = \text{VaR}_\gamma(.)$ and $\rho^\Pi(.) = \text{VaR}_\alpha(.)$, the corresponding distortion functions are $\Gamma(u) = 1_{[\gamma,1]}(u)$ and $\Pi(u) = 1_{[\alpha,1]}(u)$. The algorithm for finding the optimal stop-loss retention level follows steps:

1. Check if $K(u, c, b) > 0$, then $a^* = 0$ while $\max \mathbb{E}[H(U_1)] = \int_0^K F_X(x)dx$; Otherwise, go to step 2.

2. Check if $F_X^{-1}(\alpha) = K$ or $F_X^{-1}(\alpha) = K + b$, then $a^* \in [\text{VaR}_{\alpha \vee \gamma}(X), \infty)$ while $\max \mathbb{E}[H(U_1)] = 0$; Otherwise, go to step 3.

3. Check if $F_X^{-1}(\gamma) = K + b$, then $a^* \in [K, \text{VaR}_{\alpha \wedge \gamma}(X)]$ while $\max \mathbb{E}[H(U_1)] = 0$; Otherwise, $a^* \in [K, \infty)$ while $\max \mathbb{E}[H(U_1)] = 0$. 

Thank you for Your Attention.