About the Role of the Dependence between Mortality and Interest Rates when pricing Guaranteed Annuity Options

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Introduction

- A large number of life insurance products such as annuities include interest rate and mortality risks.
- Nowadays, it is widely admitted that mortality intensities behave in a stochastic way. Some references: Milevsky and Promislow [2001], Dahl [2004], Biffis [2005], Dahl and Moller [2006], Dahl et al. [2008].
- Even when stochastic mortality models showed up during the 90's, the actuarial community made the assumption that mortality risk is independent of interest risk. The assumption that interest risk and mortality risk are independent may seem acceptable in the short term. However, in the long term, it seems intuitive that demographic changes can affect the economy. Several papers show that interest risk and mortality risk are not necessarily independent.

Several papers show that interest risk and mortality risk are not necessarily independent:

- Nicolini [2004]) shows that the increase in adult life expectancy in the 17th and 18th century can be considered a key factor in explaining the increase in the accumulation of assets and the decline in the interest rate that took place in pre-industrial England. In particular, he shows that this independence relation is not maintained in general.
- Favero et al. [2011] investigate the probability that the slowly evolving mean in the log-dividend price ratio is related to some demographic trends,
- Maurer [2014] explores how demographic changes affect the value of financial assets.
- Dacorogna and Cadena [2015] is especially interested in the dependence between mortality and market risks in periods of extremes, such as significant pandemic outbreak. Their data samples do not contain such outbreaks and therefore they pick the worst 10 years of mortality out of a dataset of 80 years. They observe a reduction of the performance of some financial variables and an increase in correlation, but it is difficult to assess statistically.

- As also suggested by Miltersen and Persson [2005], Jalen and Mamon [2009], Liu et al. [2014], it might be reasonable to have a pricing framework allowing for a dependence between mortality and interest rates.
- Correlation risk should be taken into account according to Solvency II, see QIS5.
- Solvency II recommends to calculate Best Estimates by using the real-world probability $\mathbb{P}$ for insurance risks like mortality risk, but this recommendation is natural for independent situations between e.g. mortality and interest rate risks. But if this is not the case?
  This paper is about pricing.
- Dhaene et al. [2013] investigate the conditions under which it is possible (or not) to transfer the independence assumption from the physical world to the pricing world. There are still a lot of open questions in a dynamic continuous-time world.
  We start our model under the risk-neutral probability.

Goal

- Miltersen and Persson [2005] take into account dependence in a stochastic forward force of mortality rate model, see e.g. Cairns et al. [2006].
- Jalen and Mamon [2009] and Liu et al. [2014] studied the dependence between the mortality and the interest rates in a Gaussian framework: increasing linear correlations imply in this model increasing prices for Guaranteed Annuity Options (GAOs).
- We investigate the consequences of other models and other dependence structures between mortality risk and interest rate risk on the pricing of insurance contingent claims such as indexed annuities and GAOs.
- We consider factor models with factors in a general affine framework like in Keller-Ressel and Mayerhofer [2015]. In particular, we look at the following models:
    - the multi-CIR model
    - the Wishart model.
- Wishart processes have been first defined by Bru [1991] and are introduced in finance by Gouriéroux and Sufana [2003, 2011]. They represent a matrix extension of the square-root model.

Gaussian model of Liu et al. (2014)

The dynamics of the force of mortality μ_t of an individual at time t with age $x + t$ are given by

$$d\mu_t = c\mu_t dt + \xi dZ_t \quad (1)$$

and the dynamics of interest rate r_t by

$$dr_t = a(b - r_t)dt + \sigma dW_t \quad (2)$$

with $corr(Z_t, W_t) = \rho$ where Z_t and W_t are standard Brownian motions.

The authors show numerically that increasing linear correlations imply in this model increasing prices for Guaranteed Annuity Options (GAOs)

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Model

We consider a filtered probability space $(\Omega, \mathcal{G}, \mathbb{Q})$ where the probability $\mathbb{Q}$ is a risk-neutral probability measure.
Let

- $\mathcal{R}_t$ be the filtration generated by the interest rate process
- $\tau_M(x)$ the random variable corresponding to the future lifetime of an individual aged x at time 0, admitting a random intensity $\mu(t, x+t)$.
- $\mathcal{M}_t$ the filtration generated by the mortality intensity process $\mu(t, x+t)$
- $\mathcal{F}_t := \mathcal{R}_t \vee \mathcal{M}_t$ the sigma-algebra generated by $\mathcal{R}_t \cup \mathcal{M}_t$
- $\mathcal{H}_t$ the smallest filtration with respect to which $\tau_M(x)$ is a stopping time:

$$\mathcal{H}_t := \sigma(1_{\{\tau_M(x) \leq s; 0 \leq s \leq t\}})$$

- $\mathcal{G}_t := \mathcal{F}_t \vee \mathcal{H}_t$

- We regard $\tau_M(x)$ as the first jump-time of a nonexplosive $\mathcal{G}$-counting process N recording at each time $t \geq 0$ whether the individual died ($N_t \neq 0$) or not ($N_t = 0$).

We further assume that

$$\mathbb{Q}\left(N_T - N_t = k \mid \mathcal{F}_T \vee \mathcal{H}_t\right) = \frac{\left(\int_t^T \mu(s, x+s)ds\right)^k \exp\left(-\int_t^T \mu(s, x+s)ds\right)}{k!}.$$

The probability of survival up to time $T \geq t$ on the set $\{\tau_M(x) > t\}$ is given by

$$\mathbb{Q}(\tau_M(x) > T|\mathcal{G}_t) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T \mu(s,x+s)ds}|\mathcal{G}_t\right].$$

In the following, if there is no confusion about the age x, we will denote just τ_M.

In this setting, μ_t is the force of mortality of an individual at time t with age $x + t$. We assume that the dynamics of the force of mortality μ_t are given by

$$\mu_t = \bar{\mu} + \langle M, X_t \rangle, \tag{3}$$

and the dynamics of interest rate r_t by

$$r_t = \bar{r} + \langle R, X_t \rangle \tag{4}$$

with X being either a multi-CIR process with state spaces $\mathbb{R}_+^m$ or an (affine) Wishart process on the state space S_d^+ (the cone of positive semidefinite symmetric $d \times d$ matrices).

Multidimensional CIR processes

We consider a 3-dimensional affine positive process, having independent components $X_t = (X_{1t}, X_{2t}, X_{3t})^T$ ruled by the dynamics

$$dX_{it} = k_i(\theta_i - X_{it})dt + \sigma_i \sqrt{X_{it}} dW_{it}$$

under $\mathbb{Q}$.
We assume that the interest rate process $(r_t)_{t \geq 0}$ and the mortality process $(\mu_t)_{t \geq 0}$ are described by

$$r_t = \bar{r} + X_{1t} + X_{2t}, \quad \mu_t = \bar{\mu} + m_2 X_{2t} + m_3 X_{3t},$$

with $\bar{r}$, $\bar{\mu}$, m_2 and m_3 constants.

So with $R = (1, 1, 0)^T$ and $M = (0, m_2, m_3)^T$:

$$r_t = \bar{r} + \langle R, X_t \rangle$$

and

$$\mu_t = \mu(x, x+t) = \bar{\mu} + \langle M, X_t \rangle,$$

with $\langle R, X_t \rangle = \sum_{i=1}^3 R_i X_i$

Survival benefit

We are interested in calculating the value at time t of an insurance contingent claim paying a $\mathcal{F}_T$-measurable single benefit C_T upon survival of the insured at time T which we denote by $SB_t(C_T;T)$ as in Biffis [2005].

Using the risk-neutral pricing approach, this basic insurance contract has the following value at time t

$$\begin{aligned} SB_t(C_T;T) &= 1_{\{\tau_M>t\}}\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T(r_s+\mu_s)ds}C_T|\mathcal{F}_t\right] \\ &= 1_{\{\tau_M>t\}}\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T\bar{r}+\bar{\mu}+\langle R+M,X_s\rangle ds}C_T|\mathcal{F}_t\right]. \end{aligned} \quad (5)$$

Change of probability measure

We will define the probability measure $\mathbb{Q}^{T,\mu}$ with the Radon-Nikodym derivative of $\mathbb{Q}^{T,\mu}$ with respect to $\mathbb{Q}$ as (see also Liu et al. [2014])

$$\frac{d\mathbb{Q}^{T,\mu}}{d\mathbb{Q}} := \zeta_T = \frac{e^{-\int_0^T \bar{r}+\bar{\mu}+\langle R+M,X_s\rangle ds}}{\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^T \bar{r}+\bar{\mu}+\langle R+M,X_s\rangle ds}\right]},$$

and we define for $t \leq T$

$$\zeta_t^T = \mathbb{E}^{\mathbb{Q}}[\zeta_T|\mathcal{F}_t]. \tag{6}$$

Therefore, using Bayes' rule, for any $\mathcal{F}_T$-measurable random variable C_T:

$$SB_t(C_T;T) \quad = \quad 1_{\{\tau_M>t\}}\tilde{P}(t,T)\mathbb{E}^{\mathbb{Q}^{T,\mu}}\left[C_T|\mathcal{F}_t\right]. \tag{7}$$

with

$$\tilde{P}(t,T) = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T \bar{r}+\bar{\mu}+\langle R+M,X_s\rangle ds}\right].$$

$\tilde{P}(t,T)$ denotes the price at time t of a pure endowment insurance. We will call it a *survival zero-coupon bond* (SZCB hereafter) with maturity T (for an insured of age x at time 0).

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Whole life annuity in the unified approach

A whole life annuity due starting at time T is an insurance product that pays out a monetary unit at each date $T, T+1, T+2, \ldots$ until the death of the insured. Therefore, the price at time T of a whole life annuity due starting at time T with yearly payments of one unit for a person aged x at time 0 is given by

$$\begin{aligned}
\ddot{a}_x(T) &= \sum_{j=0}^{\omega-(x+T)} \tilde{P}(T, T+j) \\
&= \sum_{j=0}^{\omega-(x+T)} \mathbb{E}_T^{\mathbb{Q}}\left[e^{-\int_T^{T+j}(\bar{r}+\bar{\mu})+\langle(R+M),X_s\rangle ds}\right] \\
&= \sum_{j=0}^{\omega-(x+T)} e^{-(\bar{r}+\bar{\mu})(j)} e^{-\Phi_{(0,v)}(j,R+M)-\langle\Psi_{(0,v)}(j,R+M),X_T\rangle}
\end{aligned}$$

where $\Phi_{(0,v)}(j, R+M)$ and $\Psi_{(0,v)}(j, R+M)$ are solutions to generalized Riccati equations as in e.g. Keller-Ressel and Mayerhofer [2015] and where ω is the largest possible survival age.

Indexed whole life annuity in the unified approach

We now consider a T_1-years deferred indexed whole life annuity due which turns out a monetary unit plus a percentage γ of the short rate at each policy date $T_1, T_1+1, T_1+2, \ldots$ upon survival of the insured. We denote this T_1-years deferred indexed annuity by $SB_0(\ddot{a}_{x+T_1}(1+\gamma r); T_1)$. Therefore

$$SB_0(\ddot{a}_{x+T_1}(1+\gamma r); T_1) = \sum_{h=T_1}^{\omega-x-1} SB_0(1+\gamma r_h; h) \quad (8)$$

$$= \sum_{h=T_1}^{\omega-x-1} \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^h (r_s+\mu_s)ds}(1+\gamma r_h)\right]. \quad (9)$$

We provide two different approaches for evaluating this indexed annuity product in the setting of this unified affine approach:

- a change of measure approach
- the Fourier method or Duffie et al. [2000] method

Proposition

The present value of a T_1-years deferred life annuity which pays out a monetary unit plus a percentage γ of the short rate at each policy date $T_1, T_1+1, T_1+2, \ldots$ upon survival of the insured is given by

$$SB_0(\ddot{a}_{x+T_1}(1+\gamma r); T_1) = \sum_{h=T_1}^{\omega-x-1} \tilde{P}(0,h)\left(1+\gamma\bar{r}+\gamma\left\langle R, \mathbb{E}^{\mathbb{Q}^{h,\mu}}[X_h]\right\rangle\right) \quad (10)$$

$$= \sum_{h=T_1}^{\omega-x-1} \Big((1+\gamma\bar{r})\tilde{P}(0,h) \quad (11)$$

$$+ \gamma e^{-(\bar{r}+\bar{\mu})h} L^0_\nu(0,h,-(R+M),0,\nu R)\Big)$$

where L^0_ν denotes the derivative wrt $\nu \in \mathbb{R}$ at $\nu = 0$ of the following function

$$L(t,T,\theta_1,\theta_2,\nu\theta_3) = \mathbb{E}^{\mathbb{Q}}_t\left[e^{\int_t^T \langle\theta_1, X_u\rangle du + \langle(\theta_2+\nu\theta_3), X_T\rangle}\right], \quad (12)$$

with $(t,T,\theta_1,\theta_2,\nu\theta_3) \in [0,T]\times\mathbb{R}_+\times\mathbb{R}^{3d}$ for which the transform is well-defined.

Guaranteed Annuity Options (GAO) in the unified approach

We consider a GAO giving to the policyholder the right to choose at time T between an annual payment of g where g is a fixed rate called the guaranteed annuity rate or a cash payment equal to the capital 1 (see e.g. Pelsser [2003], Liu et al. [2013, 2014] and Zhu and Bauer [2011]).
At time T the value of the GAO is given by

$$\begin{aligned} V(T) &= \max(g\ddot{a}_x(T), 1) \\ &= 1 + g \max\left(\ddot{a}_x(T) - \frac{1}{g}, 0\right). \end{aligned}$$

Applying the risk neutral valuation procedure, we can write the value of the optional part of a GAO entered by an x-year policyholder at time $t = 0$ as

$$C(0, x, T) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^T (r_s+\mu_s)ds} g \max\left(\ddot{a}_x(T) - \frac{1}{g}, 0\right)\right]; \qquad (13)$$

and when using the probability measure $\mathbb{Q}^{T,\mu}$ defined in (6), by

$$C(0, x, T) = g\tilde{P}(0, T)\mathbb{E}^{\mathbb{Q}^{T,\mu}}\left[\max\left(\ddot{a}_x(T) - \frac{1}{g}, 0\right)\right]. \qquad (14)$$

$\longrightarrow$ Monte-Carlo simulations since similar to basket options.

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Multidimensional CIR processes

In this subsection, we model X by an n-dimensional affine process whose independent components evolve according to the CIR risk neutral dynamics

$$dX_{it} = k_i(\theta_i - X_{it})dt + \sigma_i \sqrt{X_{it}} dW_{it}^{\mathbb{Q}}, \quad i = 1, .., n. \tag{15}$$

The price of the SZCB $\tilde{P}(t, T)$ can be easily derived in this framework:

$$\begin{aligned} \tilde{P}(t, T) &= \mathbb{E}_t^{\mathbb{Q}} \left[e^{-\int_t^T (\bar{r}+\bar{\mu}) + \langle (R+M), X_s \rangle ds} \right] \\ &= e^{-(\bar{r}+\bar{\mu})(T-t)} \prod_{i=1}^{n} \mathbb{E}_t^{\mathbb{Q}} \left[e^{-\int_t^T (R_i+M_i) X_{is} ds} \right] \\ &= e^{-(\bar{r}+\bar{\mu})(T-t)} \prod_{i=1}^{n} e^{-\phi_i(T-t, R_i+M_i) - \psi_i(T-t, R_i+M_i) X_{it}} \end{aligned} \tag{16}$$

where ϕ_i and ψ_i are solution of the Riccati equations (Duffie et al. [2000]).

Indeed, ϕ_i and ψ_i are the following solutions of the Riccati equations (Duffie et al. [2000]).

$$\psi_i(\tau, u_i) = \frac{2u_i}{\zeta(u_i)+k_i} - \frac{4u_i\zeta(u_i)}{\zeta(u_i)+k_i}\frac{1}{(\zeta(u_i)+k_i)\exp[\zeta(u_i)\tau]+\zeta(u_i)-k_i}, \quad (17)$$

$$\phi_i(\tau, u_i) = -\frac{k_i\theta_i}{\sigma_i^2}[\zeta(u_i)+k_i]\tau + \frac{2k_i\theta_i}{\sigma_i^2}\log[(\zeta(u_i)+k_i)\exp[\zeta(u_i)\tau]+\zeta(u_i)-k_i] - \frac{2k_i\theta_i}{\sigma_i^2}\log(2\zeta(u_i)), \quad (18)$$

where $\zeta(u_i) = \sqrt{k_i^2 + 2u_i\sigma_i^2}$.

Indexed annuity in the multi-CIR case

Proposition (Change of measure approach)

In the multidimensional CIR model, the present value of a T_1-years deferred life annuity which pays a monetary unit plus a percentage γ of the short rate at each policy date $T_1, T_1+1, T_1+2, \ldots$ upon survival of the insured can be written as follows:

$$SB_0(\ddot{a}_{x+T_1}(1+\gamma r); T_1) = \sum_{h=T_1}^{\omega-x-1} \tilde{P}(0,h)\left(1+\gamma\bar{r}+\gamma\sum_{i=0}^{n} R^i \mathbb{E}^{\mathbb{Q}^{h,\mu}}[X_{ih}]\right)$$

where the expectation of X_{ih} under the measure $\mathbb{Q}^{h,\mu}$ is given by

$$\mathbb{E}^{\mathbb{Q}^{h,\mu}}[X_{ih}] = \left(x_i e^{\frac{\sigma_i^2}{k_i\theta_i u_i}\phi_i(h,u_i)} + k_i\theta_i \int_0^h \exp\left(\frac{\sigma_i^2}{k_i\theta_i u_i}\phi_i(h-s,u_i)+k_i s\right) ds\right) \times \exp(-k_i h).$$

Proposition (Fourier approach)

In the multidimensional CIR model, the present value of a T_1-years deferred life annuity which pays a monetary unit plus a percentage γ of the short rate at each policy date $T_1, T_1+1, T_1+2, \ldots$ upon survival of the insured is given by

$$\begin{aligned} SB_0(\ddot{a}_{x+T_1}(1+\gamma r);\, T_1) &= \sum_{h=T_1}^{\omega-x-1} \Big((1+\gamma\bar{r})\tilde{P}(0,h) \\ &\qquad + \gamma e^{-(\bar{r}+\bar{\mu})h} L^0_\nu(0,h,-(R+M),0,\nu R) \Big) \end{aligned}$$

where

$$L^0_\nu(t,T,\theta_1,\theta_2,\nu\theta_3) = \left[\sum_{i=1}^{n} \partial_\nu \widetilde{L}_i(t,T,\theta_{i1},\theta_{i2},\nu\theta_{i3}) \prod_{\substack{j=1\\ j\neq i}}^{n} \widetilde{L}_j(t,T,\theta_{j1},\theta_{j2},\nu\theta_{j3}) \right] \Bigg|_{\nu=0},$$

with

$$\widetilde{L}_k(t,T,\theta_{k1},\theta_{k2},\nu\theta_{k3}) = \mathbb{E}_t^{\mathbb{Q}} \left[e^{\int_t^T \theta_{k1} X_{ku} du + (\theta_{k2}+\nu\theta_{k3}) X_{kT}} \right].$$

The Wishart Case

In this section we assume that the affine process $(X_t)_{t\geq 0}$ is a d-dimensional Wishart process. Given a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{Q})$ and a $d \times d$ matrix Brownian motion W (i.e. a matrix whose entries are independent Brownian motions), the Wishart process X_t (without jumps) is defined as the solution of the $d \times d$-dimensional stochastic differential equation

$$dX_t = (\beta Q^\top Q + HX_t + X_t H^\top)dt + \sqrt{X_t} dW_t Q + Q^\top dW_t^T \sqrt{X_t}, \quad t \geq 0,$$

where $X_0 = x \in S_d^+$, $\beta \geq d - 1$, $H \in M_d$ (the set of real $d \times d$ matrices) and $Q \in GL_d$ (the set of invertible real $d \times d$ matrices). In the case where $\beta \geq d + 1$, the process takes values in S_d^{++}, i.e. the interior of the cone of positive semidefinite symmetric $d \times d$ matrices denoted by S_d^+ (see e.g. Cuchiero et al. [2011] and Da Fonseca et al. [2013]).

The price of a SZCB in the Wishart case can be derived as follows

$$\begin{aligned} \tilde{P}(t,T) &= \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T \bar{r}+\bar{\mu}+\mathrm{Tr}((R+M)X_s)ds}\right] \\ &= e^{-(\bar{r}+\bar{\mu})(T-t)}e^{-\phi(T-t,R+M)-\mathrm{Tr}[\psi(T-t,R+M)X_t]}, \end{aligned} \quad (19)$$

where ϕ and ψ solve the following system of ODE's ($\tau = T-t$)

$$\begin{cases} \frac{\partial\phi}{\partial\tau} = Tr[\beta Q^\top Q\psi(\tau,R+M)], \\ \phi(0,R+M)=0, \\ \frac{\partial\psi}{\partial\tau} = \psi(\tau,R+M)H + H^\top\psi(\tau,R+M) - 2\psi(\tau,R+M)Q^\top Q\psi(\tau,R+M) + R+M, \\ \psi(0,R+M)=0. \end{cases}$$

As proposed in Grasselli and Tebaldi [2008] and Da Fonseca et al. [2008], matrix Riccati equations can be linearized:

$$\begin{pmatrix} A_{11}(\tau) & A_{12}(\tau) \\ A_{21}(\tau) & A_{22}(\tau) \end{pmatrix} = \exp\left(\tau\begin{pmatrix} H & 2Q^\top Q \\ R+M & -H^\top \end{pmatrix}\right)$$

It turns out that

$$\begin{aligned} \psi(\tau,R+M) &= A_{22}^{-1}(\tau)A_{21}(\tau) \\ \phi(\tau,R+M) &= \frac{\beta}{2}\left(\log(\det(A_{22}(\tau))) + \tau Tr[H^\top]\right). \end{aligned}$$

We first derive the dynamics of the Wishart process under the measure $\mathbb{Q}^{T,\mu}$. To do that, we find the dynamics of the SZCB price $\tilde{P}(t,T)$ that can be found by applying the Ito's lemma to the expression (19):

$$\begin{aligned}\frac{d\tilde{P}(t,T)}{\tilde{P}(t,T)} &= (\bar{r}+\bar{\mu}-\text{Tr}[(R+M)X_t])dt - \text{Tr}[\psi(\tau,R+M)\sqrt{X_t}dW_tQ] \\ &\quad - \text{Tr}[\psi(\tau,R+M)Q^\top(dW_t)^\top\sqrt{X_t}].\end{aligned}$$

Girsanov's theorem gives the link between the Brownian motions under $\mathbb{Q}^{T,\mu}$ and $\mathbb{Q}$:

$$dW_t^{\mathbb{Q}^{T,\mu}} = dW_t^{\mathbb{Q}} + \sqrt{X_t}\psi(T-t,R+M)Q^\top dt.$$

As a consequence, the dynamics of X_t under $\mathbb{Q}^{T,\mu}$ are given by

$$\begin{aligned}dX_t &= \beta Q^\top Qdt + \left(H - Q^\top Q\psi(\tau,R+M)\right)X_tdt + X_t\left(H^\top - \psi(\tau,R+M)Q^\top Q\right) \\ &\quad + \sqrt{X_t}\left(dW_t^{\mathbb{Q}^{T,\mu}}\right)Q + Q^\top\left(dW_t^{\mathbb{Q}^{T,\mu}}\right)^\top\sqrt{X_t}.\end{aligned}$$

Using the distribution of a Wishart process with time-varying linear drift (see Kang and Kang [2013]) leads to a first expression of the indexed annuity.

Proposition (Change of measure approach)

In the Wishart model, the present value of a T_1-years deferred life annuity which pays out a monetary unit plus a percentage γ of the short rate at each policy date $T_1, T_1+1, T_1+2, \ldots$ upon survival of the insured is given by

$$SB_0(\ddot{a}_{x+T_1}(1+\gamma r); T_1) = \sum_{h=T_1}^{\omega-x-1} \tilde{P}(0,h)\left(1+\gamma\bar{r}+\gamma n\,\mathrm{Tr}[RV(0)]+\gamma\,\mathrm{Tr}\left[R\tilde{\Psi}(0)^\top x\tilde{\Psi}(0)\right]\right),$$

where $V(t)$ and $\tilde{\Psi}(t)$ are solutions of the following system of ordinary differential equations

$$\begin{aligned} \frac{d}{dt}\tilde{\Psi}(t) &= -\widetilde{H}(h-t,R+M)^\top\tilde{\Psi}(t), \\ \frac{d}{dt}V(t) &= -\tilde{\Psi}(t)^\top Q^\top Q\tilde{\Psi}(t), \end{aligned}$$

with $\widetilde{H}(h-t,R+M) = H - Q^\top Q\psi(h-t,R+M)$ and terminal conditions $\tilde{\Psi}(T) = I_d$ and $V(T) = 0$.

Proposition (Fourier approach)

In the Wishart model, the present value of a T_1-years deferred life annuity which pays a monetary unit plus a percentage γ of the short rate at each policy date $T_1, T_1+1, T_1+2, \ldots$ upon survival of the insured is given by

$$SB_0(\ddot{a}_{x+t}(1+\gamma r); T_1) = \sum_{h=T_1}^{\omega-x-1} \Big((1+\gamma\bar{r})\tilde{P}(0,h) + \gamma e^{-(\bar{r}+\bar{\mu})h}\left(\mathrm{Tr}(a^0_\nu(h)X_0) + c^0_\nu(h)\right) e^{\mathrm{Tr}(a^0(h)X_0)+c^0(h)} \Big)$$

where (see e.g. Chiarella et al. [2014])

$$\begin{cases} a^0(h) = A_{22}(h)^{-1}A_{21}(h) \\ c^0(h) = -\frac{1}{2}\,\mathrm{Tr}\left[(Q^\top Q)^{-1}\beta Q^2 \log(A_{22}(h))\right] - \frac{h}{2}\,\mathrm{Tr}\left[(Q^\top Q)^{-1}\beta Q^2 H^\top\right] \\ a^0_\nu(h) = -(A_{22}(h))^{-1}RA_{12}(h)a^0(h) + (A_{22}(h))^{-1}RA_{11}(h) \\ c^0_\nu(h) = -\frac{1}{2}\,\mathrm{Tr}\left(\beta D_{\log,\, A_{22}(h)}(RA_{12}(h))\right) \end{cases}$$

with

$$\begin{pmatrix} A_{11}(h) & A_{12}(h) \\ A_{21}(h) & A_{22}(h) \end{pmatrix} = \exp\left(h \begin{pmatrix} H & 2Q^\top Q \\ R+M & -H^\top \end{pmatrix}\right),$$

and where $D_{\log,\, A}(E)$ represents the Fréchet derivative of the logarithm function computed for the matrix A in the direction $E \in M_n$.

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Multidimensional CIR processes

We consider a 3-dimensional affine positive process, having independent components $X_t = (X_{1t}, X_{2t}, X_{3t})$ ruled by the dynamics

$$dX_{it} = k_i(\theta_i - X_{it})dt + \sigma_i \sqrt{X_{it}} dW_{it}$$

under $\mathbb{Q}$.
We assume that the interest rate process $(r_t)_{t \geq 0}$ and the mortality process $(\mu_t)_{t \geq 0}$ are described by

$$r_t = \bar{r} + X_{1t} + X_{2t}, \quad \mu_t = \bar{\mu} + m_2 X_{2t} + m_3 X_{3t},$$

with $\bar{r}$, $\bar{\mu}$, m_2 and m_3 constants.
In our illustration the coefficient m_2 is fixed in the experiments and m_3 is chosen such that

$$\mathbb{E}^{\mathbb{Q}}[\mu_T] = C_x(T), \tag{21}$$

meaning that the expectation of the mortality is fixed to a level $C_x(T)$ corresponding to the mortality rate, predicted by the Gompertz-Makeham model at age $x + T$ for an individual aged x at time 0.

The linear pairwise correlation between $(r_t)_{t\geq 0}$ and $(\mu_t)_{t\geq 0}$, denoted by ρ_t, is given by

$$\rho_t = \frac{m_2\sigma_2^2 X_{2t}}{\sqrt{\sigma_1^2 X_{1t} + \sigma_2^2 X_{2t}}\sqrt{m_2^2\sigma_2^2 X_{2t} + m_3^2\sigma_3^2 X_{3t}}}. \tag{22}$$

The parameters of the insurance products are given in the following table, with $x = 50$ and $\omega = 100$. We choose $\bar{r} = -0.12332$ and $\bar{\mu} = 0$. The expected value in (21) is fixed to the level $C_{50}(15) = 0.014$.
Parameter values of the interest rate process, see Chiarella et al. [2016].

CIR process	Parameters			
X_1	$k_1 = 0.3731$	$\theta_1 = 0.074484$	$\sigma_1 = 0.0452$	Initial value : 0.0510234
X_2	$k_2 = 0.011$	$\theta_2 = 0.245455$	$\sigma_2 = 0.0368$	Initial value : 0.0890707
X_3	$k_3 = 0.01$	$\theta_3 = 0.0013$	$\sigma_3 = 0.0015$	Initial value : 0.0004

Table: Parameter values of the 3-dimensional CIR process

Product	Parameters	
GAO	$g = 0.111$	$T = 15$
Indexed annuity	$\gamma = 0.06$	$T_1 = 15$

Table: Parameter values of the insurance contracts.

Sensitivity study with respect to the correlation between mortality and interest rates in the multi-CIR model

m_2	ρ_0	GAO with formula (13)	GAO with formula (14)	Indexed annuity
-0.1	-0.7196	0.2246406 (0.0008418)	0.2257942 (0.0005775)	5.8269507
-0.01	-0.4128	0.2531518 (0.0009962)	0.2531801 (0.0006618)	6.1072984
-0.001	-0.0516	0.2567771 (0.0010047)	0.2571203 (0.0006748)	6.1387679
0.001	0.0520	0.2579532 (0.0010129)	0.2588907 (0.0006766)	6.1458521
0.01	0.4355	0.2638415 (0.0010419)	0.2611032 (0.0006890)	6.1781468
0.1	0.7310	0.3000678 (0.0012543)	0.3003570 (0.0008096)	6.5415269

Table: Fair values for the GAO and the indexed annuity.

- Increasing linear correlation implies increasing prices (like in the Gaussian framework).

Wishart model

We assume that X follows a Wishart process. We recall that the mortality process $(\mu_t)_{t\geq 0}$ and the interest rate process $(r_t)_{t\geq 0}$ are modeled by

$$\begin{aligned} r_t &= \bar{r} + \langle R, X_t \rangle = \bar{r} + \mathrm{Tr}(RX_t), \\ \mu_t &= \bar{\mu} + \langle M, X_t \rangle = \bar{\mu} + \mathrm{Tr}(MX_t), \qquad t \geq 0. \end{aligned}$$

In the following, we will make the particular choice of R and M :

$$R = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad M = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \tag{23}$$

and this in order to treat the most simply possible case. For this choice, the stochastic correlation between $(r_t)_{t\geq 0}$ and $(\mu_t)_{t\geq 0}$ is given by

$$\rho_t = \frac{(Q_{11}Q_{12} + Q_{22}Q_{21})X_t^{12}}{\sqrt{(Q_{11}^2 + Q_{21}^2)X_t^{11}(Q_{22}^2 + Q_{12}^2)X_t^{22}}}. \tag{24}$$

Example 1

In this experiment, we consider the following Wishart process:

$$H = \begin{pmatrix} -0.5 & 0.4 \\ 0.007 & -0.008 \end{pmatrix}, \quad Q = \begin{pmatrix} 0.06 & -0.0006 \\ -0.06 & 0.006 \end{pmatrix},$$

$$X_0 = \begin{pmatrix} 0.01 & X_0^{12} \\ X_0^{12} & 0.001 \end{pmatrix}, \quad \beta = 3, \qquad \bar{r} = 0.04, \qquad \bar{\mu} = 0.$$

X_0^{12}	ρ_0	GAO with formula (13)	GAO with formula (14)	Indexed annuity
-0.002	0.4894936	0.2448621 (0.0003981)	0.2451137 (0.0002435)	5.7801950
-0.0015	0.3671202	0.2437137 (0.0004092)	0.2443471 (0.0002408)	5.7729164
-0.0005	0.1223734	0.2436714 (0.0004018)	0.2437706 (0.0002430)	5.7583871
0	0	0.2431196 (0.0004078)	0.2435689 (0.0002410)	5.7511364
0.0005	-0.1223734	0.2424844 (0.0004001)	0.2429534 (0.0002398)	5.7438950
0.0015	-0.3671202	0.2412104 (0.0004056)	0.2420545 (0.0002440)	5.7294398
0.002	-0.4894936	0.2411214 (0.0004041)	0.2417495 (0.0002440)	5.7222261

Table: Fair values for the GAO and the indexed annuity.

- Increasing linear correlation implies increasing prices.

Example 2

In this second experiment, we consider the following Wishart process:

$$H = \begin{pmatrix} -0.5 & 0.4 \\ 0.007 & -0.008 \end{pmatrix}, \quad Q = \begin{pmatrix} 0.06 & 0.0006 \\ 0.06 & 0.006 \end{pmatrix},$$

$$X_0 = \begin{pmatrix} 0.01 & X_0^{12} \\ X_0^{12} & 0.001 \end{pmatrix}, \quad \beta = 3, \qquad \bar{r} = 0.04, \qquad \bar{\mu} = 0.$$

X_0^{12}	ρ_0	GAO with formula (13)	GAO with formula (14)	Indexed annuity
-0.002	-0.4894936	0.1994176 (0.0005877)	0.1993275 (0.0003667)	5.2104471
-0.0015	-0.3671202	0.1990714 (0.0005945)	0.1987619 (0.0003767)	5.2045963
-0.0005	-0.1223734	0.1988364 (0.0006011)	0.1986171 (0.0003681)	5.1929144
0	0	0.1984553 (0.0005948)	0.1977835 (0.0003701)	5.1870834
0.0005	0.1223734	0.1984125 (0.0005943)	0.1976614 (0.0003675)	5.1812590
0.0015	0.3671202	0.1982640 (0.0005990)	0.1969242 (0.0003690)	5.1696300
0.002	0.4894936	0.1979702 (0.0005998)	0.1964036 (0.0003824)	5.1638254

Table: Fair values for the GAO and the indexed annuity.

- Increasing linear correlation implies decreasing prices.

Example 3

In this third experiment, we consider the following Wishart process:

$$H = \begin{pmatrix} -0.5 & 0.4 \\ 0.007 & -0.008 \end{pmatrix}, \quad Q = \begin{pmatrix} 0.06 & Q_{12} \\ Q_{12} & 0.006 \end{pmatrix}, \quad X_0 = \begin{pmatrix} 0.01 & 0.001 \\ 0.001 & 0.001 \end{pmatrix}, \quad \beta = 3,$$

Q_{12}	ρ_0	GAO with formula (13)	GAO with formula (14)	Indexed annuity
-0.01	-0.2942210	0.2952542 (0.0008533)	0.2953898 (0.0007196)	6.6586982
-0.006	-0.2447468	0.3385183 (0.0006100)	0.3373131 (0.0005179)	7.0908734
-0.002	-0.1099389	0.3511130 (0.0005080)	0.3512829 (0.0003793)	7.1946104
0.002	0.1099389	0.3296504 (0.0006325)	0.3285171 (0.0004363)	6.9353738
0.006	0.2447468	0.2799801 (0.0008396)	0.2788112 (0.0006115)	6.3815167
0.01	0.2942210	0.2176668 (0.0010351)	0.2159984 (0.0007818)	5.6571110

Table: Fair values for the GAO and the indexed annuity in the Wishart specification, Example 3.

- No monotone relation between correlation and prices (unlike the previous cases).

Plan

- The value of a GAO cannot always be explained only in terms of the initial pairwise linear correlation, e.g. not in advanced affine models (such as the Wishart one).
- It is clear that the dependence between mortality and interest rates has an implication on the pricing of insurance products and that several behaviors are possible.

Thank you for your attention

E. Biffis. Affine processes for dynamic mortality and actuarial valuations. *Insurance: Mathematics and Economics*, 37(3):443–468, 2005.

M.-F. Bru. Wishart processes. *Journal of Theoretical Probability*, 4:725–751, 1991.

A.C.G. Cairns, D. Blake, and K. Dowd. Pricing death: Frameworks for the valuation and securitisation of mortality risk. *Astin Bulletin*, 36:79–120, 2006.

C. Chiarella, J. Da Fonseca, and M. Grasselli. Pricing range notes within Wishart affine models. *Insurance: Mathematics and Economics*, 58:193–203, 2014.

C. Cuchiero, D. Filipovic, E. Mayerhofer, and J. Teichmann. Affine processes on positive semidefinite matrices. *The Annals of Applied Probability*, 21(2): 397–463, 2011.

J. Da Fonseca, M. Grasselli, and C. Tebaldi. A Multifactor Volatility Heston Model. *Quantitative Finance*, 8(6):591–604, 2008.

J. Da Fonseca, A. Gnoatto, and M. Grasselli. A flexible matrix Libor model with smiles. *Journal of Economic Dynamics and Control*, 37(4):774–793, 2013.

M. Dacorogna and M. Cadena. Exploring the dependence between mortality and market risks. *SCOR Papers*, 2015.

M. Dahl. Stochastic mortality in life insurance: market reserves and mortality-linked insurance contracts. *Insurance: Mathematics and Economics*, 35(1):113–136, 2004.

M. Dahl and T. Moller. Valuation and hedging of life insurance liabilities with systematic mortality risk. *Insurance: Mathematics and Economics*, 39(2): 193–217, 2006.

M. Dahl, M. Melchior, and T. Moller. On systematic mortality risk and risk-minimization with survivor swaps. *Scandinavian Actuarial Journal*, 2-3: 114–146, 2008.

J. Dhaene, A. Kukush, E. Luciano, W. Schoutens, and B. Stassen. On the (in-)dependence between financial and actuarial risks. *Insurance: Mathematics and Economics*, 52(3):522–531, 2013.

D. Duffie, J. Pan, and K. Singleton. Transform analysis and asset pricing for affine jump-diffusions. *Econometrica*, 68(6):1343–1376, 2000.

C.A. Favero, A.E. Gozluklu, and A. Tamoni. Demographic trends, the dividend-price ratio, and the predictability of long-run stock market returns. *Journal of Financial and Quantitative Analysis*, 46(5):1493–1520, 2011.

C. Gouriéroux and R. Sufana. Wishart Quadratic Term Structure Models. Working paper, available on SSRN at http://papers.ssrn.com/sol3/papers.cfm?abstract_id=757307, 2003.

C. Gouriéroux and R. Sufana. Discrete time Wishart term structure models. *Journal of Economic Dynamics and Control*, 35(6):815–824, 2011.

M. Grasselli and C. Tebaldi. Solvable Affine Term Structure Models. *Mathematical Finance*, 18:135–153, 2008.

L. Jalen and R. Mamon. Valuation of contingent claims with mortality and interest rate risks. *Math. Comput. Model.*, 49:1893–1904, 2009.

C. Kang and W. Kang. Exact Simulation of Wishart Multidimensional Stochastic Volatility Model. Working paper, 2013. URL http://arxiv.org/abs/1309.0557v1.

M. Keller-Ressel and E. Mayerhofer. Exponential Moments of Affine Processes. *Annals of Applied Probability*, 25(2):714–752, 2015.

X. Liu, R. Mamon, and H. Gao. A comonotonicity-based valuation method for guaranteed annuity options. *J. Computational Applied Mathematics*, 250: 58–69, 2013.

X. Liu, R. Mamon, and H. Gao. A generalized pricing framework addressing correlated mortality and interest risks: a change of probability measure approach. *Stochastics An International Journal of Probability and Stochastic Processes*, 86(4):594–608, 2014.

T.A. Maurer. Asset pricing implications of demographic change. Working paper, 2014.

M. Milevsky and S. Promislow. Mortality derivatives and the option to annuitise. *Insurance: Mathematics and Economics*, 29(3):299–318., 2001.

K. Miltersen and S.-A. Persson. Is mortality dead? stochastic forward force of mortality rate determined by no arbitrage. *working paper*, 2005.

E. Nicolini. Mortality, interest rates, investment, and agricultural production in 18th century england. *Explorations in Economic History*, 41(2):130–155, 2004.

A. Pelsser. Pricing and hedging guaranteed annuity options via static option replication. *Insurance: Mathematics and Economics*, 33:283–296, 2003.

N. Zhu and D. Bauer. Applications of forward mortality factor models in life insurance practice. *The Geneva Papers on Risk and Insurance*, 36:567–594, 2011.