Distributions with Heavy Tails in Orlicz Spaces

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1. Introduction.

The main object of this work is the asymptotic calculation of the ruin probability in the renewal risk model under regular varying claims in the case of Orlicz spaces, whence the premium payment rate and the initial capital satisfy some stability conditions.
Taking into account the fact that Orlicz spaces are Banach lattices, we point out that the *Kantorovich functional* (see the definition in (1)) with respect to a *quasi-interior point* of some positive cone is well-defined.

We call *Young function* any convex, even, continuous function Φ satisfing the relations $\Phi(0) = 0$, $\Phi(-x) = \Phi(x) \geq 0$ and

$$\lim_{x \to \infty} \Phi(x) = \infty\,.$$

The *conjugate function* of Φ is given by

$$\Psi(y) = \sup_{x \geq 0} \{xy - \Phi(x)\}\,, \qquad \forall\, y \geq 0\,.$$

We denote by L^Φ the following linear space named after Orlicz

$$\left\{X \in L^0(\Omega,\, \mathcal{F},\, \mathbb{P}) \mid \mathbb{E}_{\mathbb{P}}[\Phi(c\, X)] < \infty, \text{ for some } c > 0\right\} .$$

The Orlicz space L^Φ admits two equivalent norms: The first one is known as *Luxemburg norm* of X,

$$\|X\|_\Phi = \inf\left\{\lambda > 0 : \mathbb{E}_{\mathbb{P}}\left[\Phi\left(\frac{X}{\lambda}\right)\right] \leq 1\right\} ,$$

and the second known as *Orlicz norm* of X is defined on L^Ψ as follows:

$$\|X\|_\Phi^* = \sup\left\{\mathbb{E}_{\mathbb{P}}[XY] \mid \|Y\|_\Psi \leq 1\right\} .$$

For some Orlicz space L^{Φ} subspace of $L^0(\Omega, \mathcal{F}, \mathbb{P})$ we consider the dual pair of Orlicz spaces, as the $\langle M^{\Phi}, L^{\Psi} \rangle$, where M^{Φ} is the Orlicz heart of the L^{Φ} defined by the Young function Φ (see further in [3, Par. 4.1]):

$$M^{\Phi} := \{X \in L^0(\Omega,\, \mathcal{F},\, \mathbb{P}) \,|\, \mathbb{E}_{\mathbb{P}}(\Phi(c\, X)) < \infty\,, \quad \forall\, c > 0\}\,.$$

2. The properties of the Young function Φ.

Definition 1. *A quasi-interior point of some partially ordered Orlicz space L^{Φ}, is any element $u \in L^{\Phi}_{+}$ for which the solid subspace*

$$I_u = \bigcup_{n=1}^{\infty} [-nu, nu] \,,$$

under the partial ordering is dense in the space L^{Φ} (with respect to the Luxemburg norm or equivalently to the Orlicz norm).

Definition 2. *Let L^{Φ} be a Banach lattice and there exist a quasi-interior point u in the positive cone L^{Φ}_{+}. We call Kantorovich functional the function*

$$f_u(v) = \inf\{y \in \mathbb{R} \mid v \leq y\,u\} \,. \tag{1}$$

Remark 3. The existence of quasi-interior point u in the positive cone L^{Φ}_{+} has an analogy to actuarial terminology. If as positive cone we consider the acceptance set for the surplus of the company, then the Net Profit Condition correspond to the requirement for existence of quasi-interior point. Indeed, the Net Profit Condition, in the form of positive safety loading, permits some restricted disturbance in the risk process, which provides the stability condition for the operation of the company. In case of zero safety loading this stability get lost and the event of ruin appears almost surely.

Theorem 1. *Let L^{Φ} be a Banach lattice and u a quasi-interior point of L^{Φ}_{+}. Then there exists a function $f_u : L^{\Phi}_{+} \to \mathbb{R}_{+}$ defined through (1), such that the functional $f_u(v)$ is additive and hence can be extended to a continuous Kantorovich functional $f_u \in (L^{\Phi})^{*} = M^{\Psi}$.*

Definition 4. [Krasnoselski] *We call N-Young function, a Young function Φ defined on $\mathbb{R}$, which satisfies the conditions:*

1)

$$\lim_{x \to 0} \frac{\Phi(x)}{x} = 0\,,$$

2)

$$\lim_{x \to \infty} \frac{\Phi(x)}{x} = \infty\,,$$

3) The relation $\Phi(x) = 0$ implies $x = 0$.

Definition 5. *We say that a Young function Φ which satisfies the Δ_2-property if there exist a constant $k > 0$ and a $x_0 \in \mathbb{R}$ such that holds*

$$\Phi(2x) \leq k\,\Phi(x)\,, \quad \forall\, x \geq x_0\,.$$

Lemma 6. *Let consider a N-Young function* Φ *with the* Δ_2*-property and suppose that* L^{Φ}_{+} *is closed. Then there exists a sequence of Kantorovich functionals* $\{f_n\}_{n\in\mathbb{N}}$*, such that*

$$L^{\Phi}_{+} \setminus \{0\} = \left\{X \in L^{\Phi}_{+} \mid f_n(X) > 0\right\} .$$

Let us consider a subspace Y (being a subspace with the Riesz Decomposition Property) of an ordered normed space E whose positive cone is defined by $\{f_n, n \in \mathbb{N}\}$ and study the convergence behaviour of the sequence $f_n(v)$, for $n \in \mathbb{N}$, when v is a quasi-interior point of L_+^Φ.

Definition 7. *The ordered subspace Y of a normed space E has the* ***Maximum Support Property*** *if each subspace F of Y, which is equal to Y or F is a solid subspace of Y generated by a nonzero element of Y_+ has the property: an element $x \in F_+$ is a quasi-interior point of F_+ if and only if $supp(x) = supp(F_+)$ (see [6, Def. 12]).*

Proposition 8. *Necessary and sufficient condition for $f_n(v) \to \infty$, where v is a quasi-interior point of L_+^Φ is that L_+^Φ has the Maximum Support Property.*

Corollary 9. *Any Orlicz space L^{Φ}, whose Φ is Δ_2-function and N-Young, satisfies the Maximum Support Property, and further its closed cone L^{Φ}_{+} has a positive basis.*

The relation between quasi-interior points and standard assumptions used in actuarial science, may be motivated as follows:

Proposition 10. *For an Orlicz space L^{ϕ}, if Φ is N-Young and Δ_2-function and the Maximum Support Property holds for any quasi-interior point x of L^{Φ}_{+}, then holds the heavy-tail condition*

$$\mathbb{E}(e^{r\,x}) = \infty\,,$$

for any $r > 0$.

3. Slow variation on Orlicz spaces.

Definition 11. *A measurable positive function* L *defined on* $\mathbb{R}_+$, *is called slowly varying with respect to a Young function* Φ *(symbolically* $L \in \mathcal{R}_{\Phi,\,0}$*), if for a certain* u *quasi-interior point of* L^{Φ}_{+}

$$\lim_{f_n(v)\to\infty} \frac{L[t\, f_n(v)]}{L[f_n(v)]} = 1\,, \tag{2}$$

for any $t > 0$, *with* $f_n(v) = \inf\{y \in \mathbb{R} \mid v \leq y\, u\}$.

Remark 12. The convergence (2) under $f_n(v) \to \infty$, is independent from the quasi-interior point $u \in L^{\Phi}_{+}$. Indeed, if $f_n(v) \to \infty$, this implies that for v large enough $f_n(v) > M$, for some $M > 0$. From the definition of $f_n(v)$, we get that $\inf\{y \in \mathbb{R} \mid v \leq y\, u\} > M$, namely $y > M$ for any such y. If we take another quasi-interior point $v \in L^{\Phi}_{+}$, then $y\, v > M\, v$. This implies that $y \notin \{y' \in \mathbb{R} \mid v \leq y'\, v\}$. Hence, $M < y < f_n(v) \leq y'$, for any such y', which implies that $f_n(v)$ belongs to the same neighborhood of ∞.

Following the line from [2] we have the following result.

Theorem 2. *If $L \in \mathcal{R}_{\Phi,\,0}$, then the convergence in (2) is uniform with respect to t on any compact subset of $\mathbb{R}_+$.*

Corollary 13. *If $L \in \mathcal{R}_{\Phi,\,0}$, then the function L is bounded over any finite interval, located right enough in $\mathbb{R}_+$. The function $g[f_n(v)] = \log L\left[e^{f_n(v)}\right]$ is also bounded over any finite interval, located right enough in $\mathbb{R}_+$.*

Thus, we obtain the following $Karamata$-type Theorem for functions from the class $\mathcal{R}_{\Phi,\,0}$, which belong to L^{Φ}.

Theorem 3. *$L \in \mathcal{R}_{\Phi,\,0}$ in the space L^{Φ} with respect to a quasi-interior point u of $L_{+}^{\Phi} = \{v \in L^{\Phi} \mid v \geq 0, \mathbb{P} - a.s.\}$, and the associated functional f_n, if and only if L may be represented in the form*

$$L[f_n(v)] = c[f_n(v)] \exp\left\{\int_{b}^{f_n(v)} \epsilon(y)\, \frac{dy}{y}\right\}, \tag{3}$$

where $f_n(v) \geq b$ for some $b \in \mathbb{R}$, the c is some real-valued measurable function, such that $c(t) \to c \in (0, \infty)$ and the ϵ is some real-valued measurable function, such that $\epsilon(t) \to 0$, as $t \to \infty$.

4. Karamata's theory on Orlicz spaces.

Definition 14. *A measurable positive function R defined on $\mathbb{R}_+$, is called regularly varying with index $-\alpha < 0$ with respect to a Young function Φ (symbolically $R \in \mathcal{R}_{\Phi,\,-\alpha}$), if for any u quasi-interior point of L^Φ_+*

$$\lim_{f_n(v)\to\infty} \frac{R[t\,f_n(v)]}{R[f_n(v)]} = t^{-\alpha}\,, \tag{4}$$

for any $t > 0$, with $f_n(v) = \inf\{y \in \mathbb{R} \mid v \leq y\,u\}$.

Theorem 4. *$R \in \mathcal{R}_{\Phi,\,-\alpha}$ with index $-\alpha < 0$, if and only if R permits the following representation*

$$R[f_n(v)] = c[f_n(v)] \exp\left\{\int_b^{f_n(v)} \epsilon(y)\,\frac{dy}{y}\right\}, \tag{5}$$

where $f_n(v) \geq b$ for some b, c is some real-valued measurable function, such that $c(w) \to c \in (0, \infty)$ and $\epsilon(w) \to -\alpha$, while $w \to \infty$.

Corollary 15. *Let R be a positive locally integrable function over the interval $[f_0, \infty)$. The following are valid*

1) If for some $\alpha \leq 1$ holds

$$f_n(v)\, R[f_n(v)] \sim (1-\alpha) \int_{f_0}^{f_n(v)} R(y)\, dy\,,$$

$f_n(v) \to \infty$, then $R \in \mathcal{R}_{\Phi,\,-\alpha}$.

2) If for some $\alpha > 1$ holds

$$f_n(v)\, R[f_n(v)] \sim (\alpha-1) \int_{f_n(v)}^{\infty} R(y)\, dy\,,$$

$f_n(v) \to \infty$, then $R \in \mathcal{R}_{\Phi,\,-\alpha}$.

Theorem 5. *Let* $L \in \mathcal{R}_{\Phi,\,0}$ *in the space* L^{Φ} *with respect to a quasi-interior point* u *of* $L^{\Phi}_{+} = \{v \in L^{\Phi} \mid v \geq 0, \mathbb{P} - a.s.\}$, *and the associated functional* f_n. *Then For any* $A > 1$ *and* $\delta > 0$, *there exists* $f_0 = f_0(A,\, \delta)$ *such that*

$$\frac{L[f_n(v)]}{L[f_n(z)]} \leq A \left(\frac{f_n(v)}{f_n(z)}\right)^{\delta} \bigvee \left(\frac{f_n(z)}{f_n(v)}\right)^{\delta}, \tag{6}$$

for any $f_n(v),\ f_n(z) \geq f_0$.
If the function L *is locally bounded from* 0 *and* ∞ *(this means bounded on every compact subset of* $[0,\, \infty)$*), then for any* $\delta > 0$ *there is some* $a_0 = a_0(\delta) > 1$, *such that*

$$\frac{L[f_n(z)]}{L[f_n(v)]} \leq a_0 \max\left\{\left(\frac{f_n(z)}{f_n(v)}\right)^{\delta}, \left(\frac{f_n(v)}{f_n(z)}\right)^{\delta}\right\}, \tag{7}$$

for any $f_n(v),\ f_n(z) > 0$.

Corollary 16. *Let $L \in \mathcal{R}_{\Phi,0}$ in the space L^Φ with respect to a quasi-interior point u of $L^\Phi_+ = \{v \in L^\Phi \mid v \geq 0, \mathbb{P} - a.s.\}$, and the associated functional f_n. If L is a locally bounded function over the interval $[f_n(z), \infty)$, then*
1) For any $\alpha \geq -1$

$$\int_{f_n(z)}^{f_n(v)} t^\alpha L(t)\, dt \sim \frac{f_n^{\alpha+1}(v)}{\alpha+1} L[f_n(v)]\,, \tag{8}$$

as $f_n(v) \to \infty$.
2) For any $\alpha < -1$

$$\int_{f_n(v)}^{\infty} t^\alpha L(t)\, dt \sim -\frac{f_n^{\alpha+1}(v)}{\alpha+1} L[f_n(v)]\,, \tag{9}$$

as $f_n(v) \to \infty$.

5. Tauberian results.

Theorem 6. *Let a non-decreasing continuous from right function R, with $R[f_n(v)] = 0$ for any $[f_n(v)] < 0$. If $L \in \mathcal{R}_{\Phi,\,0}\,,\; c \geq 0\,,\; \alpha \geq 0$, then the following are equivalent*
1)

$$R[f_n(v)] \sim \frac{c\, f_n^{\alpha}(v)}{\Gamma(\alpha+1)}\, L[f_n(v)]\,, \tag{10}$$

as $f_n(v) \to \infty$.
2)

$$\widehat{R}(s) := \int_0^{\infty} e^{-s\,x}\, R(dx) \sim \frac{c}{s^{\alpha}}\, L\left(\frac{1}{s}\right)\,, \tag{11}$$

as $s \downarrow 0$.
When $c = 0$, the first part is interpreted as $R[f_n(v)] = f_n^{\alpha}(v)\, L[f_n(v)]$, as $f_n(v) \to \infty$.
In case $c > 0$ from the first or second part we obtain

$$R[f_n(v)] \sim \frac{1}{\Gamma(\alpha+1)}\, \widehat{R}\left(\frac{1}{f_n(v)}\right),$$

as $f_n(v) \to \infty$.

Lemma 17. *Let* $L \in \mathcal{R}_{\Phi,\,0}$. *For any constant* $\sigma > 0$ *and any* $f_n(v) \geq w$, *for the function* $\widetilde{L}[f_n(v)]$ *defined by*

$$\widetilde{L}[f_n(v)] = f_n^{\sigma}(v) \sup_{w \leq t \leq f_n(v)} t^{-\sigma}\, L(t)\,,$$

holds $\widetilde{L}[f_n(v)] \sim L[f_n(v)]$ *as* $f_n(v) \to \infty$.

Now we study the asymptotic behavior of the integral Lebesgue in the form

$$\int_a^b f(t)\, L(f_n(v)\, t)\, dt\,, \tag{12}$$

with L some slowly varying function and $0 < a < b < \infty$. For some function $f(t)$ integrable over the interval $[a,\, b]$, considering the

$$\int_a^b f(t)\, \left[\frac{L(f_n(v)\, t)}{L[f_n(v)]} - 1\right]\, dt\,,$$

from Theorem 2 about the uniform convergence, we see that

$$\int_a^b f(t)\, L(f_n(v)\, t)\, dt \sim L[f_n(v)]\, \int_a^b f(t)\, dt\,, \tag{13}$$

as $f_n(v) \to \infty$.

Theorem 7. *Let L slowly varying function and with some constant $\sigma > 0$ and some function f there exists the integral*

$$\int_a^{\infty} t^{\sigma} f(t)\, dt < \infty\,, \tag{14}$$

for some $a > 0$. Then there exists also

$$\int_a^{\infty} f(t)\, L(f_n(v)\, t)\, dt < \infty\,,$$

and holds

$$\int_a^{\infty} f(t)\, L(f_n(v)\, t)\, dt \sim L[f_n(v)] \int_a^{\infty} f(t)\, dt\,,$$

as $f_n(v) \to \infty$.

6. Asymptotic results on regular variation.

Proposition 18. *Let* $\{X_n, n = 1, 2, ...\}$ *a sequence of i.i.d. random variables having the distribution* F*, where we suppose that* $\overline{F} \in \mathcal{R}_{\Phi,-\alpha}$*. Then, the sum* $A(w) = \sum_{k=1}^{\infty} w_k X_k$ *converges* $\mathbb{P}$*-a.s. if one of the following two conditions holds:*
1)

$$\sum_{k=1}^{\infty} w_k^{\eta} < \infty,$$

some $\eta < \alpha$*, with* $\eta \leq 1$*.*
2) $\mathbb{E}(X) = 0$ *and the sum* $\sum_{k=1} w_k^{\eta}$ *converges for some* $\eta < \alpha$*, where* $\eta \leq 2$ *(or for* $\eta = 1$*, when* $\alpha = 1$*).*

Lemma 19. *If* $\overline{F} \in \mathcal{R}_{\Phi,-\alpha}$*, while moreover we assume that*

$$\sum_{k=1}^{\infty} w_k^{\eta} < \infty\,,$$

for some $\eta < \alpha$*, then we obtain:*

$$\lim_{f_n(v)\to\infty} \frac{1}{\mathbb{P}[f_n(X_1) > f_n(v)]} \sum_{k=1}^{\infty} \mathbb{P}[w_k X_k > f_n(v)] = \sum_{k=1}^{\infty} w_k^{\alpha}\,,$$

$$\lim_{f_n(v)\to\infty} \frac{1 - \prod_{k=1}^{\infty} \mathbb{P}[w_k X_k \leq f_n(v)]}{\mathbb{P}[f_n(X_1) > f_n(v)]} = \sum_{k=1}^{\infty} w_k^{\alpha}.$$

Lemma 20. *If $\overline{F} \in \mathcal{R}_{\Phi,\,-\alpha}$ then*

$$\mathbb{P}\left[\sum_{k=1}^{n} w_k X_k > f_n(v)\right] \in \mathcal{R}_{\Phi,\,-\alpha}\,,$$

$$\mathbb{P}\left[\bigvee_{k=1}^{n} w_k X_k > f_n(v)\right] \in \mathcal{R}_{\Phi,\,-\alpha}\,,$$

and the asymptotic relations

$$\lim_{f_n(v)\to\infty} \frac{\mathbb{P}[\sum_{k=1}^{n} w_k X_k > f_n(v)]}{\mathbb{P}[f_n(X_1) > f_n(v)]} = \lim_{f_n(v)\to\infty} \frac{\mathbb{P}[\bigvee_{k=1}^{n} w_k X_k > f_n(v)]}{\mathbb{P}[f_n(X_1) > f_n(v)]} = \sum_{k=1}^{n} w_k^{\alpha}\,.$$

Theorem 8. *Let* $\{X_n, n = 1, 2, ...\}$ *a sequence of i.i.d. random variables with common distribution* F *for which* $\overline{F} \in \mathcal{R}_{\Phi,-\alpha}$, $\alpha > 0$. *We assume the existence of a sequence of positive real numbers* $\{w_n, n = 1, 2, ...\}$, *such that*

$$\sum_{n=1}^{\infty} w_k^{\eta} < \infty\,,$$

for some $\eta < 1 \wedge \alpha$. *Then the following asymptotic relations hold:*

$$\mathbb{P}\left(\bigvee_{k=1}^{\infty} w_k f_n(X_k) > f_n(v)\right) \quad \sim \quad \overline{F}[f_n(v)] \sum_{k=1}^{\infty} w_k^{\alpha}\,, \tag{15}$$

$$\mathbb{P}\left(\sum_{k=1}^{\infty} w_k f_n(X_k) > f_n(v)\right) \quad \sim \quad \overline{F}[f_n(v)] \sum_{k=1}^{\infty} w_k^{\alpha}\,. \tag{16}$$

We consider now the case of the corresponding sum having random weights (not necessarily positive) $\{\beta_n, n = 1, 2, ...\}$ which constitute a sequence of i.i.d. random variables, being independent from $\{X_n, n = 1, 2, ...\}$. This sum is denoted as follows:

$$A(\beta) = \sum_{k=1}^{\infty} \beta_k f_n(X_k).$$

Lemma 21. *We assume that* $\overline{F} \in \mathcal{R}_{\Phi,-\alpha}$*, where* $\alpha > 0$*, and the random weights* $\{\beta_n, n = 1, 2, ..\}$ *are i.i.d. random variables, independent from* $\{X_n, n = 1, 2, ...\}$*. Let consider the following conditions:*
1) For $\alpha < 1$*, there exists some* $\epsilon \in (0, \alpha)$*, such that*

$$\sum_{k=1}^{\infty} \mathbb{E}(\beta_k^{\alpha+\epsilon} + \beta_k^{\alpha-\epsilon}) < \infty\,,$$

and $\alpha + \epsilon < 1$*.*
2) For $\alpha \geq 1$*, there exists some* $\epsilon > 0$*, such that*

$$\sum_{k=1}^{\infty} \left[\mathbb{E}(\beta_k^{\alpha+\epsilon} + \beta_k^{\alpha-\epsilon})\right]^{1/(\alpha+\epsilon)} < \infty\,.$$

Then, the asymptotic relation holds:

$$\mathbb{P}[A(\beta) > f_n(v)] \sim \overline{F}[f_n(v)] \sum_{k=1}^{\infty} \mathbb{E}(\beta_k^{\alpha}) \, .$$

Let now consider the renewal risk model with constant ingerest rate $r \geq 0$. We assume that the surplus $\{U_r(t)\,,\ t \geq 0\}$, satisfies the stochastic differential equation

$$U_r(dt) = c\,dt + U_r(t)\,r\,dt - S(dt)\,, \tag{17}$$

with

$$S(t) = \sum_{i=1}^{N(t)} Z_i\,,$$

the aggregate claim up to time t. This gives after multiplication with factor $e^{-r\,t}$, under the initial condition $U_r(0) = u$ the equation

$$U_r(t) = ue^{r\,t} + c\,\frac{e^{r\,t} - 1}{r} - \int_0^t e^{r(t-y)}\,S(dy)\,.$$

We introduce the present value of the surplus $U_r(t)$ at the initial moment

$$V_r(t) := e^{-rt} U_r(t) = u + c\,\frac{1-e^{-rt}}{r} - \int_0^t e^{-ry}\,S(dy)\,. \tag{18}$$

Now we define the ruin probability under constant interest rate $\psi_r(u)$ as follows:

$$\psi_r(u) := \left[\bigcup_{t\geq 0} \{V_r(t) < 0\} \;\middle|\; V_r(0) = u \right] = \mathbf{P}\left[\inf_{t\geq 0} V_r(t) < 0 \;\middle|\; V_r(0) = u \right].$$

By Lemma 21, we may extract an asymptotic estimate for the ruin probability in the renewal model.

Theorem 9. *We assume the renewal risk model, where the claim payment distribution* $B \in \mathcal{R}_{\Phi,-\alpha}$*, for* $\alpha > 0$*. Then if* $f_n(v)$ *denotes the initial capital of the insurance company,*

$$\psi[f_n(v)] \sim \frac{\mathbb{E}(e^{-r\theta_1})}{1 - \mathbb{E}(e^{-r\theta_1})} \overline{B}[f_n(v)] \,.$$

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Thank you!